PART A

Unit - IV

Partial Differential Equations (PDE)

4.1 Introduction

Many problems in vibration of strings, heat conduction, electrostatics involve two or more variables. Analysis of these problems leads to partial derivatives and equations involving them. In this unit we first discuss the formation of PDE analogous to that of formation of ODE. Later we discuss some methods of solving PDE.

4.2 Definitions

An equation involving one or more partial derivatives of a function of two or more variables is called a **partial differential equation.**

The **order** of a PDE is the order of the highest derivative and the **degree** of the PDE is the degree of highest order derivative after clearing the equation of fractional powers.

A PDE is said to be **linear** if it is of first degree in the dependent variable and its partial derivatives.

If each term of the PDE contains either the dependent variable or one of its partial derivatives, the PDE is said to be **homogeneous** . Otherwise it is said to be a **nonhomogeneous** PDE.

Examples

1. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ [order = 1, degree = 1, homogeneous PDE]

2. $\frac{\partial^2 z}{\partial x \, \partial y} = x y$ [order = 2, degree = 1, nonhomogeneous PDE]

3. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$ [order = 2, degree = 1, homogeneous PDE]

4. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ [order = 2, degree = 1, homogeneous PDE]

4.3 Formation of PDE by eliminating arbitrary constants and arbitrary functions

Given a relation of the form $f(x, y, z, a, b) = 0$ where z is a function of x, y and a, b are arbitrary constants, we differentiate the given relation w.r.t x and y partially and eliminate the arbitrary constants a, b to form the PDE. In case the number of arbitrary constants are more than the number of independent variables we need appropriate number of partial derivatives of second and higher order also.

Suppose z is a function of two arbitrary functions we have to find partial derivatives upto the second order and use the necessary partial derivatives of the second order to form the PDE by eliminating the arbitrary functions.

Note : *The following standard notations when z is a function of two independent variables x, y will be used.*

$$p = \frac{\partial z}{\partial x} = z_x \quad q = \frac{\partial z}{\partial y} = z_y \quad r = \frac{\partial^2 z}{\partial x^2} = z_{xx} \quad s = \frac{\partial^2 z}{\partial x\, \partial y} = z_{xy}$$

$$t = \frac{\partial^2 z}{\partial y^2} = z_{yy}$$

WORKED PROBLEMS

Form the PDE by eliminating the arbitrary constants in the following

1. $z = (x+a)(y+b)$

>> By data, $z = (x+a)(y+b)$ ...(1)

Differentiating partially w.r.t x and y,

$$p = \frac{\partial z}{\partial x} = (y+b) \quad \text{...(2)}$$

$$q = \frac{\partial z}{\partial y} = (x+a) \quad \text{...(3)}$$

Using (2) and (3) in (1) we obtain $z = pq$

Thus $\boldsymbol{z = pq}$ is the required PDE

---

2. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

>> By data, $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ ...(1)

Differentiating (1) partially w.r.t x and y,

$$2\,\frac{\partial z}{\partial x} = 2p = \frac{2x}{a^2} \quad \text{or} \quad a^2 = \frac{x}{p} \qquad \ldots (2)$$

$$2\,\frac{\partial z}{\partial y} = 2q = \frac{2y}{b^2} \quad \text{or} \quad b^2 = \frac{y}{q} \qquad \ldots (3)$$

Using (2) and (3) in (1) we obtain

$$2\,z = x^2 \cdot \frac{p}{x} + y^2 \cdot \frac{q}{y}$$

Thus $\mathbf{2\,z = p\,x + q\,y}$ is the required PDE.

---

3. $ax^2 + by^2 + z^2 = 1$

>> By data, $z^2 = 1 - ax^2 - by^2$ ... (1)

Differentiating (1) w.r.t x and y partially,

$$2\,z\,\frac{\partial z}{\partial x} = 2\,z\,p = -2ax \quad \therefore \quad a = -\frac{z\,p}{x} \qquad \ldots (2)$$

$$2\,z\,\frac{\partial z}{\partial y} = 2\,z\,q = -2by \quad \therefore \quad b = -\frac{z\,q}{y} \qquad \ldots (3)$$

Using (2) and (3) in (1) we obtain,

$$z^2 = 1 + \frac{z\,p}{x}\,x^2 + \frac{z\,q}{y}\,y^2$$

Thus $\mathbf{(z^2 - 1) = z\,(p\,x + q\,y)}$ is the required PDE.

---

4. $z = a\,\log\,(x^2 + y^2) + b$

>> By data, $z = a\,\log\,(x^2 + y^2) + b$ ... (1)

$$\therefore \quad \frac{\partial z}{\partial x} = p = \frac{a}{x^2 + y^2}\,2x \quad \text{or} \quad p = \frac{2ax}{x^2 + y^2} \qquad \ldots (2)$$

$$\frac{\partial z}{\partial y} = q = \frac{a}{x^2 + y^2}\,2y \quad \text{or} \quad q = \frac{2\,ay}{x^2 + y^2} \qquad \ldots (3)$$

Dividing (2) by (3) we get $\frac{p}{q} = \frac{x}{y}$

Thus $\mathbf{p\,y - q\,x = 0}$ is the required PDE.

---

5. *Find the PDE of the family of all spheres whose centres lie on the plane $z = 0$ and have a constant radius 'r'.*

>> The co-ordinates of the centre of the sphere can be taken as $(a, b, 0)$ where a and b are arbitrary. r is the constant radius.

The equation of the sphere is given by

$$(x-a)^2+(y-b)^2+(z-0)^2 = r^2$$

i.e., $$(x-a)^2+(y-b)^2+z^2 = r^2 \quad \ldots(1)$$

Here a and b are arbitrary constants and have to be eliminated.

Differentiating (1) w.r.t x, y partially,

$$2(x-a)+2zp = 0 \text{ and } 2(y-b)+2zq = 0$$

∴ Dividing these equations by 2 we obtain

$$(x-a) = -zp \text{ and } (y-b) = -zq$$

Substituting these in (1) we get, $(-zp)^2+(-zq)^2+z^2 = r^2$

Thus $z^2 (p^2+q^2+1) = r^2$ is the required PDE.

---

6. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 1$

>> By data, $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \quad \ldots(1)$$

Differentiating (1) partially w.r.t x and y,

$$\frac{2x}{a^2}+\frac{2z}{c^2}p = 0 \text{ and } \frac{2y}{b^2}+\frac{2z}{c^2}q = 0$$

i.e., $$\frac{x}{a^2}+\frac{zp}{c^2} = 0 \quad \ldots(2)$$

$$\frac{y}{b^2}+\frac{zq}{c^2} = 0 \quad \ldots(3)$$

Differentiating (2) w.r.t x partially again, we get

$$\frac{1}{a^2}+\frac{1}{c^2}(zr+p^2) = 0 \quad \ldots(4)$$

∵ $$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = r$$

Now from (2), $\frac{x}{a^2} = -\frac{zp}{c^2}$ or $\frac{1}{a^2} = \frac{-zp}{c^2 x}$

Substituting this in (4) we get,

$$\frac{-zp}{c^2 x} = \frac{-1}{c^2}(zr + p^2) \quad \text{or} \quad zp = x(zr + p^2)$$

Thus $z\frac{\partial z}{\partial x} = xz\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2$ is the required PDE.

Note : *Differentiating (3) w.r.t y and using the expression for* $1/b^2$ *we can also obtain the PDE as* $z\frac{\partial z}{\partial y} = yz\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2$

---

7. $z = xy + y\sqrt{x^2 - a^2} + b$

>> By data, $z = xy + y\sqrt{x^2 - a^2} + b$ ... (1)

Differentiating (1) w.r.t x and y partially,

$$\frac{\partial z}{\partial x} = p = y + y \cdot \frac{1}{2\sqrt{x^2 - a^2}} 2x = y + \frac{xy}{\sqrt{x^2 - a^2}} \quad \ldots (2)$$

and $\frac{\partial z}{\partial y} = q = x + \sqrt{x^2 - a^2}$ ... (3)

Now from (3) $(q - x) = \sqrt{x^2 - a^2}$ and using this in (2) we have,

$$p = y + \frac{xy}{q - x}$$

i.e., $(p - y) = \frac{xy}{q - x}$ or $(p - y)(q - x) = xy$

Thus $pq = xp + yq$ is the required PDE.

Form the PDE by eliminating the arbitrary functions in the following.

8. $z = f(x^2 + y^2)$

>> By data, $z = f(x^2 + y^2)$ ... (1)

Differentiating (1) partially w.r..t x and y, we have

$$\frac{\partial z}{\partial x} = p = f'(x^2 + y^2) \cdot 2x \quad \ldots (2)$$

$$\frac{\partial z}{\partial y} = q = f'(x^2 + y^2) \cdot 2y \quad \ldots (3)$$

Dividing (2) by (3) we have $\frac{p}{q} = \frac{x}{y}$ or $py = qx$

Thus $\boldsymbol{py - qx = 0}$ is the required PDE.

---

9. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

>> By data, $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ ...(1)

Differentiating partially w.r.t x and y we have,

$$\frac{\partial z}{\partial x} = p = 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{-1}{x^2}\right)$$

$$\frac{\partial z}{\partial y} = q = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

i.e., $px^2 = -2f'\left(\frac{1}{x} + \log y\right)$ ...(2)

$(q - 2y)y = 2f'\left(\frac{1}{x} + \log y\right)$ ...(3)

Now dividing (2) by (3) we have

$$\frac{px^2}{(q-2y)y} = -1 \quad \text{or} \quad px^2 = -qy + 2y^2$$

Thus $\boldsymbol{px^2 + qy = 2y^2}$ is the required PDE.

---

10. $z = e^y f(x+y)$

>> By data $z = e^y f(x+y)$ ...(1)

$\therefore$ $\frac{\partial z}{\partial x} = p = e^y f'(x+y)$ ...(2)

$$\frac{\partial z}{\partial y} = q = e^y f'(x+y) + f(x+y)e^y$$

i.e., $q = e^y f'(x+y) + z$

or $q - z = e^y f'(x+y)$ ...(3)

Dividing (2) by (3) we obtain

$$\frac{p}{q-z} = 1 \quad \text{or} \quad p = q - z$$

Thus $\boldsymbol{p + z = q}$ is the required PDE.

---

11. $z = e^{ax+by} f(ax - by)$

>> By data, $z = e^{ax+by} f(ax-by)$ ... (1)

$$\frac{\partial z}{\partial x} = p = e^{ax+by} f'(ax-by) \cdot a + a e^{ax+by} f(ax-by)$$

$$\frac{\partial z}{\partial y} = q = e^{ax+by} f'(ax-by) \cdot (-b) + b e^{ax+by} f(ax-by)$$

i.e., $p = a e^{ax+by} f'(ax-by) + az$ ... (2)

$q = -b e^{ax+by} f'(ax-by) + bz$ ... (3)

Multiplying (2) by b, (3) by a and adding we get $bp + aq = 2abz$

Thus $\boldsymbol{b \dfrac{\partial z}{\partial x} + a \dfrac{\partial z}{\partial y} = 2abz}$ is the required PDE.

---

12. $lx + my + nz = \phi(x^2 + y^2 + z^2)$

>> By data, $lx + my + nz = \phi(x^2 + y^2 + z^2)$

Differentiating partially w.r.t x and also w.r.t y we have,

$l + np = \phi'(x^2 + y^2 + z^2) \cdot (2x + 2zp)$ ... (1)

$m + nq = \phi'(x^2 + y^2 + z^2) \cdot (2y + 2zq)$ ... (2)

Dividing (1) by (2) we get,

$$\frac{l + np}{m + nq} = \frac{x + zp}{y + zq}$$

i.e., $(x + zp)(m + nq) = (y + zq)(l + np)$

We shall multiply and simplify this equation.

Thus $\boldsymbol{(mz - ny)p + (nx - lz)q = ly - mx}$ is the required PDE.

Note : *To form the PDE from $\phi(u, v) = 0$ where u and v are functions of x, y, z, (z being a function of x, y) we proceed as follows.*

By data $\phi(u, v) = 0$ ... (1)

Differentiating (1) w.r.t x and y partially by applying chain rule we obtain

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0 \quad \ldots (2)$$

and $$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = 0 \quad \ldots (3)$$

Transferring the second term in both (2) and (3) onto the R.H.S and dividing we obtain the required PDE.

---

13. $\phi\,(x+y+z,\ x^2+y^2-z^2) = 0$

>> We have by data

$$\phi\,(u, v) = 0 \qquad \ldots(1)$$

where $u = x+y+z$ and $v = x^2+y^2-z^2$

Now, $\dfrac{\partial u}{\partial x} = 1 + \dfrac{\partial z}{\partial x} = 1+p$; $\dfrac{\partial v}{\partial x} = 2x-2z\dfrac{\partial z}{\partial x} = 2(x-zp)$

$\dfrac{\partial u}{\partial y} = 1 + \dfrac{\partial z}{\partial y} = 1+q$; $\dfrac{\partial v}{\partial y} = 2y-2z\dfrac{\partial z}{\partial y} = 2(y-zq)$

Let us differentiate (1) w.r.t x and y by applying chain rule.

i.e., $$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0 \quad \text{or} \quad \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} \qquad \ldots(2)$$

and $$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} \qquad \ldots(3)$$

Dividing (2) by (3) we obtain,

$$\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y}$$

i.e., $\dfrac{1+p}{1+q} = \dfrac{2(x-zp)}{2(y-zq)}$ or $(1+p)(y-zq) = (1+q)(x-zp)$

i.e., $y-zq+py-pqz = x-zp+qx-pqz$

i.e., $py+pz-qx-qz = x-y$

Thus $\boldsymbol{p(y+z)-q(x+z) = x-y}$ is the required PDE.

---

14. $\phi\,(xy+z^2,\ x+y+z) = 0$

>> We have by data $\phi\,(u, v) = 0 \qquad \ldots(1)$

where $u = xy+z^2$ and $v = x+y+z$

Now, $\dfrac{\partial u}{\partial x} = y+2zp$; $\dfrac{\partial v}{\partial x} = 1+p$

$\dfrac{\partial u}{\partial y} = x+2zq$; $\dfrac{\partial v}{\partial y} = 1+q$

Let us differentiate (1) w.r.t x and y, by applying chain rule.

i.e., $$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0 \quad \text{or} \quad \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} \qquad \ldots(2)$$

and $\dfrac{\partial\phi}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial\phi}{\partial v}\dfrac{\partial v}{\partial y}=0$ or $\dfrac{\partial\phi}{\partial u}\dfrac{\partial u}{\partial y}=-\dfrac{\partial\phi}{\partial v}\dfrac{\partial v}{\partial y}$ . . . (3)

Dividing (2) by (3) we obtain,

$$\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}$$

i.e., $\dfrac{y+2zp}{x+2zq}=\dfrac{1+p}{1+q}$ or $(1+p)(x+2zq)=(1+q)(y+2zp)$

i.e., $x+2zq+px+2pqz=y+2zp+qy+2pqz$

Thus $\mathbf{p(x-2z)-q(y-2z)+(x-y)=0}$ is the required PDE.

---

15. $f(x^2+2yz,\ y^2+2zx)=0$

>> We have by data,

$$f(u,v)=0 \qquad \ldots(1)$$

where $u=x^2+2yz$ and $v=y^2+2zx$ . . . (2)

Differentiating (1) partially w.r.t x and y by applying chain rule we have,

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=0 \quad\text{or}\quad \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}=-\frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \qquad \ldots(3)$$

and $\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y}=0$ or $\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y}=-\dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y}$ . . . (4)

Dividing (3) by (4) we obtain,

$$\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y} \qquad \ldots(5)$$

From (2) we obtain,

$$\frac{\partial u}{\partial x}=2x+2yp \quad ; \quad \frac{\partial v}{\partial x}=2(z+xp)$$

$$\frac{\partial u}{\partial y}=2(yq+z) \quad ; \quad \frac{\partial v}{\partial y}=2y+2xq$$

Hence (5) becomes,

$$\frac{2(x+yp)}{2(yq+z)}=\frac{2(z+xp)}{2(y+xq)}$$

or $xy+x^2q+y^2p+xypq=yzq+xypq+z^2+xzp$

Thus $\mathbf{(y^2-zx)p+(x^2-yz)q=(z^2-xy)}$ is the required PDE.

---

16. $f\left(\dfrac{xy}{z},\ z\right) = 0$

>> This problem is similar to the earlier three problems. Proceeding on the same lines the required PDE is

$$\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} \quad \ldots(1)$$

where $u = \dfrac{xy}{z}$ and $v = z$.

Hence we have

$$\frac{\partial u}{\partial x} = y\left[\frac{z - xp}{z^2}\right] \ ; \ \frac{\partial v}{\partial x} = p \ ; \ \frac{\partial u}{\partial y} = x\left[\frac{z - yq}{z^2}\right] \ ; \ \frac{\partial v}{\partial y} = q$$

Substituting these in (1) we have, $\dfrac{y(z - xp)}{x(z - yq)} = \dfrac{p}{q}$

i.e., $xp(z - yq) = yq\,(z - xp)$

Thus $\boldsymbol{xp = yq}$ is the required PDE.

---

17. $z = yf(x) + x\phi(y)$

>> By data, $z = yf(x) + x\phi(y)$ $\quad \ldots(1)$

(It should be noted that there are two arbitrary functions and hence we need the second order partial derivatives also.)

Differentiating w.r.t x and y partially,

$$\frac{\partial z}{\partial x} = p = yf'(x) + \phi(y) \quad \ldots(2)$$

$$\frac{\partial z}{\partial y} = q = f(x) + x\phi'(y) \quad \ldots(3)$$

$$\frac{\partial^2 z}{\partial x^2} = r = y f''(x) \quad \ldots(4)$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = s = f'(x) + \phi'(y) \quad \ldots(5)$$

$$\frac{\partial^2 z}{\partial y^2} = t = x\phi''(y) \quad \ldots(6)$$

Now from (2), $\dfrac{p - \phi(y)}{y} = f'(x)$ and from (3), $\dfrac{q - f(x)}{x} = \phi'(y)$

Using these in (5) we get,

$$s = \frac{p-\phi(y)}{y} + \frac{q-f(x)}{x} \quad \text{or} \quad s = \frac{px - x\phi(y) + qy - yf(x)}{xy}$$

$$\therefore \quad xys = px + qy - [x\phi(y) + yf(x)]$$

Using (1) in R.H.S we get,

$$xys = px + qy - z \quad \text{or} \quad xys + z = px + qy$$

Thus $xy \dfrac{\partial^2 z}{\partial x \partial y} + z = x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y}$ is the required PDE.

---

18. $z = x f_1(x+t) + f_2(x+t)$

>> By data, $z = x f_1(x+t) + f_2(x+t)$ . . . (1)

Differentiating partially w.r.t x and t, we have

$$\frac{\partial z}{\partial x} = x f_1'(x+t) + f_1(x+t) + f_2'(x+t)$$

$$\frac{\partial z}{\partial t} = x f_1'(x+t) + f_2'(x+t)$$

$$\frac{\partial^2 z}{\partial x^2} = x f_1''(x+t) + 2f_1'(x+t) + f_2''(x+t) \quad \ldots (2)$$

$$\frac{\partial^2 z}{\partial x \partial t} = x f_1''(x+t) + f_1'(x+t) + f_2''(x+t) \quad \ldots (3)$$

$$\frac{\partial^2 z}{\partial t^2} = x f_1''(x+t) + f_2''(x+t) \quad \ldots (4)$$

Now using the R.H.S of (4) in (2) as well as in (3) we get,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2} + 2f_1'(x+t) \quad \ldots (5)$$

$$\frac{\partial^2 z}{\partial x \partial t} = \frac{\partial^2 z}{\partial t^2} + f_1'(x+t) \quad \ldots (6)$$

Multiplying (6) by 2 and subtracting from (5) we get

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial t} = -\frac{\partial^2 z}{\partial t^2}$$

Thus $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial t} + \dfrac{\partial^2 z}{\partial t^2} = 0$ is the required PDE.

---

19. $z = f(y+x) + g(y+2x)$

>> By data, $z = f(y+x) + g(y+2x)$

$$p = \frac{\partial z}{\partial x} = f'(y+x) + 2g'(y+2x)$$

$$q = \frac{\partial z}{\partial y} = f'(y+x) + g'(y+2x)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(y+x) + 4g''(y+2x) \qquad \ldots(1)$$

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = f''(y+x) + 2g''(y+2x) \qquad \ldots(2)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''(y+x) + g''(y+2x) \qquad \ldots(3)$$

(1) – (2) will give us, $r - s = 2g''(y+2x)$ $\qquad \ldots(4)$

(2) – (3) will give us, $s - t = g''(y+2x)$ $\qquad \ldots(5)$

Now dividing (4) by (5) we get,

$$\frac{r-s}{s-t} = 2 \quad \text{or} \quad r - s = 2(s-t) \quad \text{or} \quad r - 3s + 2t = 0$$

Thus $\mathbf{\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 0}$ is the required PDE.

---

20. $z = f_1(y-2x) + f_2(2y-x)$

>> By data, $z = f_1(y-2x) + f_2(2y-x)$

$$p = \frac{\partial z}{\partial x} = -2f_1'(y-2x) - f_2'(2y-x)$$

$$q = \frac{\partial z}{\partial y} = f_1'(y-2x) + 2f_2'(2y-x)$$

$$r = \frac{\partial^2 z}{\partial x^2} = 4f_1''(y-2x) + f_2''(2y-x) \qquad \ldots(1)$$

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = -2f_1''(y-2x) - 2f_2''(2y-x) \qquad \ldots(2)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f_1''(y-2x) + 4f_2''(2y-x) \quad \ldots (3)$$

(1) $\times$ 2 + (2) will give us, $2r + s = 6f_1''(y-2x)$ ... (4)

(2) $\times$ 2 + (3) will give us $2s + t = -3f_1''(y-2x)$ ... (5)

Now dividing (4) by (5) we get,

$$\frac{2r+s}{2s+t} = -2 \quad \text{or} \quad 2r + 5s + 2t = 0$$

Thus $2\dfrac{\partial^2 z}{\partial x^2} + 5\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 0$ is the required PDE.

EXERCISES

Form the PDE by eliminating the arbitrary constants in the following (1 to 3)

1. $z = ax + by + ab$ **2.** $z = (x-a)^2 + (y-b)^2$

3. $u = ax + by + cz$

4. Find the PDE of the family of spheres having their centers on z-axis

Form the PDE by eliminating the arbitrary functions in the following (5 to 6)

5. $z = x + y + f(xy)$ **6.** $xyz = f(x+y+z)$

7. $z = f\left(xy/z\right)$ **8.** $z = (x+y)\, f(x^2 - y^2)$

9. $f(x^2 + y^2,\ z - xy) = 0$ **10.** $f(x^2 - xy,\ x/z) = 0$

11. $f(z/x^3,\ y/x) = 0$ **12.** $z = f(x) + e^y g(x)$

13. $f(x+y+z,\ x^2 + y^2 + z^2) = 0$ **14.** $z = f_1(y+2x) + f_2(y-3x)$

15. $z = f(x+iy) + \phi(x-iy)$ **16.** $z = f(x+ct) + g(x-ct)$

ANSWERS

1. $z = px + qy + pq$ **2.** $p^2 + q^2 = 4z$

3. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial y} = u$ **4.** $py = qx$

5. $px - qy = x - y$ **6.** $x(y-z)p + y(z-x)q = z(x-y)$

7. $px = qy$ **8.** $z = xq + yp$

9. $py - qx = y^2 - x^2$

10. $x^2 p + (2z^2 - xy) q = xz$

11. $px + qy = 3z$

12. $t = q$

13. $p(y-z) + q(z-x) = (x-y)$

14. $r + s - 6t = 0$

15. $r + t = 0$

16. $z_{tt} = c^2 z_{xx}$

4.4 Solution of PDE

A solution or integral of a partial differential equation is a relation between the dependent and independent variables satisfying the equation. It is important to note that the same PDE can have many independent solutions. But the solution of an ordinary differential equation is unique in the sense that the solution differs only by a constant. In this article we discuss solutions of partial differential equations of first order and first degree in the form

$$f(x, y, z, p, q) = 0 \qquad \ldots (1)$$

(1) Complete solution (integral)

Suppose $f(x, y, z, a, b) = 0$ ... (2)

is a relation from which the PDE (1) is obtained by eliminating the arbitrary constants a, b then (2) is called a **Complete solution** (integral) of the PDE represented by (1).

Referring to Problem - 1 it may be observed that we obtained the PDE $z = pq$ by eliminating a and b from the relation $z = (x+a)(y+b)$. Therefore we can say that $z = (x+a)(y+b)$ is a complete solution of the PDE $pq = z$

(2) Particular solution (integral)

A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a **particular solution** (integral) of the PDE.

For example $z = (x+2)(y+3)$ is a particular solution of the PDE $pq = z$

(3) General solution (integral)

Consider the complete solution $F(x, y, z, a, b) = 0$. Suppose we assume an arbitrary relation of the form $b = g(a)$ then we have

$$F\big(x, y, z, a, g(a)\big) = 0 \qquad \ldots (3)$$

Differentiating partially w.r.t a we get

$$\frac{\partial F}{\partial a} + \frac{\partial F}{\partial g}\frac{\partial g}{\partial a} = 0 \qquad \ldots (4)$$

Suppose it is possible to eliminate 'a' from (3) and (4), the relation so obtained is called a **general solution** of the PDE (1).

Referring to Problem - 1 again, we have $z = (x+a)(y+b)$ and the related PDE is $pq = z$

Suppose that $b = 2a$. Then we have,

$z - xy - 2ax - ay - 2a^2 = 0$. Differentiating partially w.r.t a we get

$$-2x - y - 4a = 0 \text{ and hence } a = -\frac{(2x+y)}{4}$$

Substituting this value in $z = (x+a)(y+b)$ where $b = 2a$ we get

$$z = \left[x - \frac{(2x+y)}{4} \right] \left[y - \frac{(2x+y)}{2} \right]$$

$$z = \frac{1}{8}(2x-y)(y-2x) \text{ or } z = -\frac{1}{8}(2x-y)^2$$

i.e., $(2x-y)^2 + 8z = 0$ is a general solution of the PDE $pq = z$.

(4) Singular solution (integral)

Let us consider the complete solution (2) of the PDE (1)

i.e., $F(x, y, z, a, b) = 0$ ... (2)

Differentiating partially w.r.t a and b we obtain

$$\frac{\partial F}{\partial a} = 0 \quad \text{and} \quad \frac{\partial F}{\partial b} = 0 \qquad \ldots (5)$$

Suppose it is possible to eliminate a and b from (2), (5) then the relation so obtained is called the **singular solution** of the PDE (1).

Referring to Problem - 1 again, the complete solution is $z - (x+a)(y+b) = 0$

Differentiating partially w.r.t a and b we obtain $y + b = 0$; $x + a = 0$. Hence $z = 0$ is the singular solution.

Geometrical meaning of various types of solution

A *complete solution* represents a two parameter family of surfaces.

A *particular solution* represents a particular surface of the family of surfaces given by the complete solution. A general solution represents the envelope of the one parameter family of surfaces. The singular solution represents the envelope of the two parameter family of surfaces.

4.5 Solution of non-homogeneous PDE by direct integration

In this method we find the dependent variable which being the solution, by removing the differential operators through the process of *anti differentiation*, that is ***integration***.

To illustrate the method, let us first consider an ordinary differential equation (ODE):

$$\frac{dy}{dx} = x^2$$

Integrating, $y = \int x^2 \, dx + k$, where k is the constant of integration.

$\therefore$ the solution is $y = \frac{x^3}{3} + k$.

On the other hand, if we consider the PDE $\frac{\partial u}{\partial x} = x^2$,

where u is a function of x and y, then $u = \int x^2 \, dx + k$, where k may be arbitrary function of y, because y has been treated as constant.

Hence the solution is represented in the form,

$$u = \frac{x^3}{3} + f(y), \text{ where } f(y) \text{ is an arbitrary function of } y.$$

We observe that the constant of integration is now an arbitrary function $f(y)$.

Therefore it is important to note that in the case of a PDE, when we integrate w.r.t. x we must add a function of y as arbitrary constant (*i.e., a function of the other independent variable or variables as the case may be*). Similarly if we integrate w.r.t. y, we must add a function of x as arbitrary constant.

Observe the following illustrative examples:

(1) If u is a function of x, y, z then the solution of $\frac{\partial u}{\partial x} = x^2$ is $u = \int x^2 \, dx + f(y, z)$

which being $u = \frac{x^3}{3} + f(y, z)$.

(2) Suppose u is a function of x and t. Then the solution of $\frac{\partial u}{\partial x} = \cos x \cos t$ is

$u = \cos t \int \cos x \, dx + f(t)$ which being $u = \cos t \sin x + f(t)$.

Similarly the solution of $\frac{\partial u}{\partial t} = \cos x \cos t$ is given by

$u = \cos x \int \cos t \, dt + f(x)$ which being $u = \cos x \sin t + f(x)$.

WORKED PROBLEMS

21. *Solve* : $\dfrac{\partial^2 u}{\partial x^2} = x + y$

>> Consider $\dfrac{\partial^2 u}{\partial x^2} = x + y$

ie., $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = x + y$

Integrating w.r.t x treating y as constant,

$$\frac{\partial u}{\partial x} = \int (x+y)\, dx + f(y) = \int x\, dx + y \int 1\, dx + f(y)$$

i.e., $$\frac{\partial u}{\partial x} = \frac{x^2}{2} + xy + f(y)$$

Integrating w.r.t x again we have,

$$u = \int \left[\frac{x^2}{2} + xy + f(y) \right] dx + g(y)$$

$$= \frac{1}{2} \int x^2\, dx + y \int x\, dx + f(y) \int 1\, dx + g(y)$$

Thus the solution is given by

$$u = \frac{x^3}{6} + \frac{x^2 y}{2} + x f(y) + g(y)$$

---

22. *Solve* : $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a.$

>> The given PDE can be written as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{x}{y} + a.$

Integrating w.r.t x treating y as constant,

$$\frac{\partial z}{\partial y} = \int \left(\frac{x}{y} + a \right) dx + f(y) = \frac{1}{y} \int x dx + a \int 1 dx + f(y)$$

i.e., $$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y)$$

Integrating now w.r.t y,

$$z = \frac{x^2}{2} \int \frac{1}{y}\, dy + a\,x \int 1 dy + \int f(y)\, dy + g(x)$$

Thus the solution is given by,

$$\mathbf{z = \frac{x^2}{2} \log y + a\,x\,y + F(y) + g(x), \text{ where } F(y) = \int f(y)\, dy}$$

---

23. *Solve* : $\dfrac{\partial^3 z}{\partial x^2\, \partial y} = \cos(2x + 3y)$

>> The given PDE can be written as

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\right) = \cos(2x + 3y)$$

Integrating w.r.t x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \int \cos(2x + 3y)\, dx + f(y)$$

i.e.,
$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\sin(2x + 3y)}{2} + f(y)$$

Again integrating w.r.t x,

$$\frac{\partial z}{\partial y} = \frac{1}{2} \int \sin(2x + 3y)\, dx + f(y) \int 1 dx + g(y)$$

i.e.,
$$\frac{\partial z}{\partial y} = \frac{-\cos(2x + 3y)}{4} + x f(y) + g(y)$$

Finally integrating w.r.t y,

$$z = -\frac{1}{4} \int \cos(2x + 3y)\, dy + x \int f(y)\, dy + \int g(y)\, dy + h(x)$$

ie.,
$$z = -\frac{1}{4}\, \frac{\sin(2x + 3y)}{3} + x F(y) + G(y) + h(x),$$

where $F(y) = \int f(y)\, dy, \quad G(y) = \int g(y)\, dy.$

Thus the solution is given by

$$\mathbf{z = -\frac{1}{12} \sin(2x + 3y) + x F(y) + G(y) + h(x)}$$

---

24. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ *for which* $\dfrac{\partial z}{\partial y} = -2 \sin y$ *when* $x = 0$, *and* $z = 0$ *if* y *is an odd multiple of* $\pi/2$. [*or* $z = 0$ *if* $y = (2n+1)\frac{\pi}{2}$]

>> Here we first find z by integration and apply the given conditions to determine the arbitrary functions occuring as constants of integration.

The given PDE can be written as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \sin x \sin y$

Integrating w.r.t x treating y as constant,

$$\frac{\partial z}{\partial y} = \sin y \int \sin x \, dx + f(y)$$

i.e., $$\frac{\partial z}{\partial y} = -\sin y \cos x + f(y) \qquad \ldots (1)$$

Integrating w.r.t y treating x as constant,

$$z = -\cos x \int \sin y \, dy + \int f(y) \, dy + g(x)$$

i.e., $$z = (-\cos x)(-\cos y) + F(y) + g(x),$$

where $F(y) = \int f(y) \, dy$.

Thus $$z = \cos x \cos y + F(y) + g(x) \qquad \ldots (2)$$

Also by data, $\dfrac{\partial z}{\partial y} = -2\sin y$ when $x = 0$. Using this in (1),

$$-2 \sin y = (-\sin y) \cdot 1 + f(y) \quad (\because \cos 0 = 1)$$

or $$f(y) = -\sin y.$$

Hence $F(y) = \int f(y) \, dy = \int -\sin y \, dy = \cos y$.

With this, (2) becomes $z = \cos x \cos y + \cos y + g(x)$ $\qquad \ldots (3)$

Using the condition that $z = 0$ if $y = (2n+1)\frac{\pi}{2}$ in (3) we have

$$0 = \cos x \cos(2n+1)\frac{\pi}{2} + \cos(2n+1)\frac{\pi}{2} + g(x).$$

But $\cos(2n+1)\frac{\pi}{2} = 0$. and hence $0 = 0 + 0 + g(x)$ or $g(x) = 0$.

Thus the solution of the PDE is given by

$$\mathbf{z = \cos x \cos y + \cos y \quad \text{or} \quad z = \cos y(\cos x + 1)}$$

---

25. *Solve* $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$ *given that* $u = 0$ *when* $t = 0$ *and* $\dfrac{\partial u}{\partial t} = 0$ *at* $x = 0$. *Also show that* $u \to \sin x$ *as* $t \to \infty$.

>> The given PDE can be written as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial t}\right) = e^{-t} \cos x$.

Integrating w.r.t x treating t as constant,

$$\frac{\partial u}{\partial t} = e^{-t} \int \cos x \, dx + f(t)$$

i.e.,
$$\frac{\partial u}{\partial t} = e^{-t} \sin x + f(t) \qquad \ldots (1)$$

Integrating w.r.t t treating x as constant,

$$u = \sin x \int e^{-t} dt + \int f(t) \, dt + g(x)$$

i.e.,
$$u = -\sin x \, e^{-t} + F(t) + g(x) \qquad \ldots (2)$$

where $F(t) = \int f(t) \, dt$

Also by data, $\dfrac{\partial u}{\partial t} = 0$ when $x = 0$.

Using this in (1), $0 = e^{-t} \sin 0 + f(t)$ and hence $f(t) = 0$

Now $F(t) = \int f(t) \, dt = \int 0 \, dt = 0$

Substituting $F(t) = 0$ in (2), we get

$$u = -e^{-t} \sin x + g(x) \qquad \ldots (3)$$

Also by data, $u = 0$ when $t = 0$

Using this in (3), $0 = -e^{0} \sin x + g(x)$ or $g(x) = \sin x$

Thus the solution is given by

$$\boldsymbol{u = -e^{-t} \sin x + \sin x \quad \text{or} \quad u = \sin x \,(1 - e^{-t})}$$

Also as $t \to \infty$, we know that $e^{-t} \to 0$. Hence $\boldsymbol{u \to \sin x}$ **as** $\boldsymbol{t \to \infty}$

---

26. *Solve* $\dfrac{\partial^2 z}{\partial x^2} = xy$ *subject to the conditions that* $\dfrac{\partial z}{\partial x} = \log(1 + y)$ *when* $x = 1$, *and* $z = 0$ *when* $x = 0$.

>> The given PDE can be written as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = xy$

Integrating w.r.t x, treating y as constant,

$$\frac{\partial z}{\partial x} = y \int x\, dx + f(y)$$

i.e., $$\frac{\partial z}{\partial x} = \frac{x^2}{2}\, y + f(y) \qquad \ldots (1)$$

Again integrating w.r.t x, treating y as constant,

$$z = \frac{y}{2} \int x^2\, dx + f(y) \int 1 dx + g(y)$$

Thus $$z = \frac{x^3 y}{6} + x f(y) + g(y) \qquad \ldots (2)$$

Also by data, $\frac{\partial z}{\partial x} = \log(1+y)$ when $x = 1$ Using this in (1),

$$\log(1+y) = \frac{1}{2}\, y + f(y) \quad \text{or} \quad f(y) = \log(1+y) - \frac{1}{2}\, y$$

With this value for $f(y)$, (2) becomes

$$z = \frac{x^3 y}{6} + x \left[\log(1+y) - \frac{1}{2}\, y \right] + g(y) \qquad \ldots (3)$$

Also by data, $z = 0$ when $x = 0$. Using this in (3)

$$0 = 0 + 0 + g(y) \quad \text{or} \quad g(y) = 0$$

Thus the solution is given by

$$z = \frac{x^3 y}{6} + x \left[\log(1+y) - \frac{1}{2}\, y \right]$$

---

27. *Solve* $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$ *subject to the conditions* $\frac{\partial z}{\partial x} = \log_e x$ *when* $y = 1$ *and* $z = 0$ *when* $x = 1$.

>> Since the condition is in terms of $\frac{\partial z}{\partial x}$, we shall use the fact that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ and write the given PDE as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{x}{y}$

Integrating w.r.t y, treating x as constant,

$$\frac{\partial z}{\partial x} = x \int \frac{1}{y}\, dy + f(x)$$

i.e., $$\frac{\partial z}{\partial x} = x \log y + f(x) \qquad \ldots (1)$$

Integrating w.r.t x, treating y as constant,

$$z = \log y \int x\, dx + \int f(x)\, dx + g(y)$$

Thus $$z = \frac{x^2}{2} \log y + F(x) + g(y) \qquad \ldots (2)$$

where $F(x) = \int f(x)\, dx$

Also by data, $\frac{\partial z}{\partial x} = \log x$ when $y = 1$. Using this in (1),

$$\log x = x \log 1 + f(x) \text{ or } f(x) = \log x, \text{ since } \log 1 = 0$$

Now $F(x) = \int f(x)\, dx = \int \log x \cdot 1\, dx = x \log x - x$, by parts.

Hence $F(x) = x \log x - x$ and we substitute this in (2).

Thus $$z = \frac{x^2}{2} \log y + x \log x - x + g(y) \qquad \ldots (3)$$

Also by data, $z = 0$ when $x = 1$. Using this in (3),

$$0 = \frac{1}{2} \log y + 0 - 1 + g(y) \quad \text{or} \quad g(y) = 1 - \frac{1}{2} \log y = 1 - \log \sqrt{y}$$

Thus the solution is given by

$$\mathbf{z = \frac{x^2}{2} \log y + x \log x - x + 1 - \log \sqrt{y}}$$

---

28. *Solve :* $\frac{\partial z}{\partial x} = 2x + y, \ \frac{\partial z}{\partial y} = x - 2y.$

>> We are given a system of equations. (*simultaneous equations*)

$$\frac{\partial z}{\partial x} = 2x + y \text{ and } \frac{\partial z}{\partial y} = x - 2y$$

Integrating w.r.t x and w.r.t y respectively,

$$z = 2 \int x\, dx + y \int 1 dx + f(y) \text{ and } z = x \int 1 dy - 2 \int y\, dy + g(x)$$

i.e., $$z = x^2 + xy + f(y) \text{ and } z = xy - y^2 + g(x)$$

We have to appropriately choose $f(y)$ and $g(x)$ to get a common expression for z. Simple comparison shows that $f(y) = -y^2$ and $g(x) = x^2$.

Thus the solution is given by $\mathbf{z = x^2 + xy - y^2}$

---

29. *Solve the system of equations,*

$$\frac{\partial u}{\partial x} = 6xy + z^3, \quad \frac{\partial u}{\partial y} = 3x^2 - z, \quad \frac{\partial u}{\partial z} = 3xz^2 - y$$

>> It is evident that u is a function of x, y, z. Integrating the given equations w.r.t x, y, z respectively, we get

$$u = 6y \int x\,dx + z^3 \int 1\,dx + f(y,z) \quad \text{or} \quad u = 3x^2 y + x z^3 + f(y,z)$$

$$u = 3x^2 \int 1\,dy - z \int 1\,dy + g(x,z) \quad \text{or} \quad u = 3x^2 y - yz + g(x,z)$$

$$u = 3x \int z^2\,dz - y \int 1\,dz + h(x,y) \quad \text{or} \quad u = x z^3 - yz + h(x,y)$$

We have to properly choose $f(y,z)$, $g(x,z)$, $h(x,y)$ to get a common expression for u. Simple comparison indicates that we must choose $f(y,z) = -yz$, $g(x,z) = xz^3$ and $h(x,y) = 3x^2 y$.

Thus the required solution is given by $u = 3x^2 y + xz^3 - yz$

---

30. *Show that the PDE* $\dfrac{\partial^2 z}{\partial t^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$ *can be reduced to the form* $\dfrac{\partial^2 z}{\partial u\,\partial v} = 0$ *using the substitution* $u = x + at$, $v = x - at$. *Hence solve the equation.*

>> We regard z as a function of u, v where u, v are functions of x and t.

$$\therefore \text{ by chain rule, } \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \& \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} \quad \ldots (1)$$

But $u = x + at$, and $v = x - at$

$$\therefore \quad \frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial x} = 1, \quad \frac{\partial u}{\partial t} = a, \quad \frac{\partial v}{\partial t} = -a$$

Substituting these in (1) we have,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{and} \quad \frac{\partial z}{\partial t} = a\frac{\partial z}{\partial u} - a\frac{\partial z}{\partial v}$$

Applying the chain rule again,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \cdot \frac{\partial v}{\partial x}$$

$$= \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v}\right) \cdot (1) + \left(\frac{\partial^2 z}{\partial v\,\partial u} + \frac{\partial^2 z}{\partial v^2}\right) \cdot (1)$$

ie., $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2} \text{ since } \frac{\partial^2 z}{\partial v\,\partial u} = \frac{\partial^2 z}{\partial u\,\partial v}$$

Also $$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial t}\right) \cdot \frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial t}\right) \cdot \frac{\partial v}{\partial t}$$

$$= \frac{\partial}{\partial u}\left(a\,\frac{\partial z}{\partial u} - a\,\frac{\partial z}{\partial v}\right) \cdot \frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(a\,\frac{\partial z}{\partial u} - a\,\frac{\partial z}{\partial v}\right) \cdot \frac{\partial v}{\partial t}$$

$$= \left(a\,\frac{\partial^2 z}{\partial u^2} - a\,\frac{\partial^2 z}{\partial u\,\partial v}\right) \cdot (a) + \left(a\,\frac{\partial^2 z}{\partial v\,\partial u} - a\,\frac{\partial^2 z}{\partial v^2}\right) \cdot (-a)$$

ie., $$\frac{\partial^2 z}{\partial t^2} = a^2\,\frac{\partial^2 z}{\partial u^2} - 2a^2\,\frac{\partial^2 z}{\partial u\,\partial v} + a^2\,\frac{\partial^2 z}{\partial v^2} \text{ since } \frac{\partial^2 z}{\partial v\,\partial u} = \frac{\partial^2 z}{\partial u\,\partial v}$$

The given PDE $\frac{\partial^2 z}{\partial t^2} = a^2\,\frac{\partial^2 z}{\partial x^2}$ now becomes,

$$a^2\left(\frac{\partial^2 z}{\partial u^2} - 2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right) = a^2\left(\frac{\partial^2 z}{\partial u^2} + 2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

ie., $$-2\,\frac{\partial^2 z}{\partial u\,\partial v} = 2\,\frac{\partial^2 z}{\partial u\,\partial v} \quad \text{or} \quad 4\,\frac{\partial^2 z}{\partial u\,\partial v} = 0$$

i.e., $$\boldsymbol{\frac{\partial^2 z}{\partial u\,\partial v} = 0}$$

Thus the equation $\frac{\partial^2 z}{\partial t^2} = a^2\,\frac{\partial^2 z}{\partial x^2}$ has become $\frac{\partial^2 z}{\partial u\,\partial v} = 0$ and we shall solve this by directintegration.

We have $\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = 0$

Integrating w.r.t u we have,

$$\frac{\partial z}{\partial v} = \text{const.} = F(v) \ (say)$$

Integrating now w.r.t v,

$$z = \int F(v)\,dv + g(u) \quad \text{or} \quad z = f(v) + g(u)$$

where $f(v) = \int F(v)\,dv$ and $u = x + at,\; v = x - at$ by data.

Thus the solution of the given PDE is represented by

$$z = f(x - at) + g(x + at)$$

4.6 Solution of homogeneous PDE involving derivatives with respect to one independent variable only

Suppose that the dependent variable has been differentiated partially w.r.t. one independent variable say x only. Then the PDE can be treated as an ordinary differential equation (ODE) and we are already familiar in solving ODE.

The arbitary constants in the solution are then replaced by arbitrary function of the other variable (y) giving a solution of the PDE.

WORKED PROBLEMS

31. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + z = 0$ *given that when* $x = 0,\; z = e^y$ *and* $\dfrac{\partial z}{\partial x} = 1.$

>> Let us suppose that z is a function of x only. The given PDE assumes the form of ODE,

$$\frac{d^2 z}{dx^2} + z = 0 \quad \text{or} \quad (D^2 + 1)\,z = 0 \quad \text{where } D = \frac{d}{dx}$$

A.E is $m^2 + 1 = 0$ or $m^2 = -1 \quad \therefore \; m = \pm\sqrt{-1} = \pm i$ (*complex roots*)
The solution of the ODE is given by

$$z = c_1 \cos x + c_2 \sin x.$$

Solution of the PDE is got by replacing c_1 and c_2 by functions of y.

Hence, solution of the PDE is given by

$$z = f(y)\cos x + g(y)\sin x \qquad \ldots (1)$$

Now we shall apply the given conditions to find $f(y)$ and $g(y)$.

By data, when $x = 0, z = e^y$. Hence (1) becomes

$$e^y = f(y)\cos 0 + g(y)\sin 0$$

ie., $\quad e^y = f(y)\cdot 1 + g(y)\cdot 0 \quad \therefore \quad f(y) = e^y$

Also by data, when $x = 0,\; \dfrac{\partial z}{\partial x} = 1.$

Differentiating (1) w.r.t. x partially we get,

$$\frac{\partial z}{\partial x} = -f(y)\ \sin x + g(y)\ \cos x$$

Applying the condition we get,

$$1 = -f(y)\ \sin 0 + g(y)\ \cos 0 \quad \therefore \quad g(y) = 1$$

We substitute $f(y) = e^y$ and $g(y) = 1$ in (1).

Thus $z = e^y \cos x + \sin x$ is the required solution.

---

32. *Solve* $\dfrac{\partial^2 z}{\partial y^2} = z$ *given that when* $y = 0,\ z = e^x$ *and* $\dfrac{\partial z}{\partial y} = e^{-x}$

>> Let us suppose that z is a function of y only.

The given PDE assumes the form of ODE,

$$\frac{d^2 z}{dy^2} = z \text{ or } \frac{d^2 z}{dy^2} - z = 0 \text{ or } (D^2 - 1)\ z = 0, \text{ where } D = \frac{d}{dy}$$

A.E is $m^2 - 1 = 0 \quad \therefore \quad m = \pm 1$ (*real and distinct roots*)

The solution of the ODE is given by

$$z = c_1 e^y + c_2 e^{-y}$$

Solution of the PDE is got by replacing c_1 and c_2 by functions of x.

Hence, solution of the PDE is given by

$$z = f(x)\ e^y + g(x)\ e^{-y} \qquad \ldots (1)$$

Now we shall apply the given conditions to find $f(x)$ and $g(x)$.

By data, when $y = 0,\ z = e^x$. Hence (1) becomes

$$e^x = f(x) + g(x)$$

Also by data, when $y = 0,\ \dfrac{\partial z}{\partial y} = e^{-x}$

Differentiating (1) w.r.t. y partially we get,

$$\frac{\partial z}{\partial y} = f(x)\ e^y - g(x)\ e^{-y}$$

Applying the condition we get,

$$e^{-x} = f(x) - g(x)$$

Now we shall solve,

$$f(x)+g(x) = e^{x} \text{ and } f(x)-g(x) = e^{-x}$$

Adding and subtracting these equations we get,

$$2f(x) = e^{x}+e^{-x} \text{ or } f(x) = \frac{e^{x}+e^{-x}}{2} = \cosh x$$

$$2g(x) = e^{x}-e^{-x} \text{ or } g(x) = \frac{e^{x}-e^{-x}}{2} = \sinh x$$

We shall substitute these in (1).

Thus $z = \cosh x \, e^{y} + \sinh x \, e^{-y}$ is the required solution.

---

33. *Solve* $\frac{\partial^2 z}{\partial x^2} = a^2 z$ *given that when* $x = 0$, $z = 0$ *and* $\frac{\partial z}{\partial x} = a \sin y$

>> Let us suppose that z is a function of x only.
The given PDE assumes the form of ODE,

$$\frac{d^2 z}{dx^2} = a^2 z \text{ or } (D^2 - a^2) z = 0 \text{ where } D = \frac{d}{dx}$$

A.E is $m^2 - a^2 = 0 \therefore m = \pm a$ (*real and distinct roots*)

The solution of the ODE is given by

$$z = c_1 e^{ax} + c_2 e^{-ax}$$

Solution of the PDE is got by replacing c_1, c_2 with functions of y.

Solution of the PDE is given by

$$z = f(y) e^{ax} + g(y) e^{-ax} \qquad \ldots (1)$$

Now we shall apply the given conditions to find $f(y)$ and $g(y)$.

By data, when $x = 0$, $z = 0$. Hence (1) becomes

$$0 = f(y) + g(y)$$

Also, when $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$.

Differentiating (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = a\, f(y)\, e^{ax} - a\, g(y)\, e^{-ax}$$

Applying the condition to this equation we get,

$$a \sin y = a f(y) - a g(y) \text{ or } \sin y = f(y) - g(y)$$

By solving $f(y) + g(y) = 0$ and $f(y) - g(y) = \sin y$ we get,

$$f(y) = \frac{\sin y}{2} \text{ and } g(y) = -\frac{\sin y}{2}$$

We shall substitute these in (1).

Now, $z = \frac{\sin y}{2} e^{ax} - \frac{\sin y}{2} e^{-ax} = \sin y \cdot \frac{e^{ax} - e^{-ax}}{2} = \sin y \sinh ax$

Thus $\boldsymbol{z = \sin y \sinh ax}$ is the required solution.

---

34. *Solve* $\frac{\partial z}{\partial x^2} = a^2 z$ *given that when* $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$ *and* $\frac{\partial z}{\partial y} = 0$.

>> This problem is same as the previous one but for a different set of conditions. Solution of the PDE is given by

$$z = f(y) e^{ax} + g(y) e^{-ax} \quad \ldots (1)$$

For the purpose of applying the given conditions we shall differentiate this w.r.t. x and y partially.

$$\frac{\partial z}{\partial x} = a f(y) e^{ax} - a g(y) e^{-ax} \quad \ldots (2)$$

$$\frac{\partial z}{\partial y} = f'(y) e^{ax} + g'(y) e^{-ax} \quad \ldots (3)$$

By data, $\frac{\partial z}{\partial x} = a \sin y$ when $x = 0$. Hence (2) becomes

$$a \sin y = a f(y) - a g(y)$$

or $$f(y) - g(y) = \sin y \quad \ldots (4)$$

Also by data, $\frac{\partial z}{\partial y} = 0$ when $x = 0$. Hence (3) becomes

$$0 = f'(y) + g'(y)$$

Integrating w.r.t. y we get,

$$f(y) + g(y) = k \quad \ldots (5)$$

where k is the constant of integration. By solving (4) and (5) simultaneously we get,

$$f(y) = \frac{1}{2}(k + \sin y) \text{ and } g(y) = \frac{1}{2}(k - \sin y)$$

We shall substitute these in (1).

Now, $z = \frac{1}{2}(k + \sin y) e^{ax} + \frac{1}{2}(k - \sin y) e^{-ax}$

or $\quad z = k \cdot \frac{1}{2} (e^{ax} + e^{-ax}) + \sin y \cdot \frac{1}{2} (e^{ax} - e^{-ax})$

Thus $\quad$ $\boldsymbol{z = k \cosh ax + \sin y \sinh ax}$ is the required solution.

---

35. *Solve* $\dfrac{\partial^2 u}{\partial x^2} + u = 0$ *where u satisfies the conditions*

$(i)\ u(0, y) = e^{1/2} \qquad (ii)\ \dfrac{\partial u}{\partial x}(0, y) = 1$

\>> Let us suppose that u is a function of x only. The given PDE assumes the form of ODE,

$$\frac{d^2 u}{dx^2} + u = 0 \text{ or } (D^2 + 1) u = 0 \text{ where } D = \frac{d}{dx}$$

A.E is $m^2 + 1 = 0$ or $m^2 = -1, \quad \therefore \quad m = \pm \sqrt{-1} = \pm i$

The solution of the ODE is given by

$$u = c_1 \cos x + c_2 \sin x$$

Solution of the PDE is got by replacing c_1 and c_2 by functions of y.

Solution of the PDE is given by

$$u(x, y) = u = f(y) \cos x + g(y) \sin x \qquad \ldots (1)$$

By data, $u(0, y) = e^{1/2} = \sqrt{e}$.

Putting $x = 0$ in (1) we have,

$$\sqrt{e} = f(y) \cos 0 + g(y) \sin 0 \qquad \therefore \quad f(y) = \sqrt{e}$$

Also by data, $\dfrac{\partial u}{\partial x}(0, y) = 1$.

Differentiating (1) w.r.t. x partially we have

$$\frac{\partial u}{\partial x} = -f(y) \sin x + g(y) \cos x$$

Putting $x = 0$ and $\dfrac{\partial u}{\partial x} = 1$ we have,

$$1 = -f(y) \sin 0 + g(y) \cos 0 \qquad \therefore \quad g(y) = 1$$

We substitute the values of $f(y)$ and $g(y)$ in (1).

Thus $\quad$ $\boldsymbol{u = \sqrt{e} \cos x + \sin x}$ is the required solution.

---

36. *Solve* $\dfrac{\partial^2 z}{\partial y^2} = z$ *given that* $z = 0$ *and* $\dfrac{\partial z}{\partial y} = \sin x$ *when* $y = 0$.

>> This problem is same as Problem-32 but for different conditions.

As in Problem-32 we have,

$$z = f(x)\, e^{y} + g(x)\, e^{-y} \qquad \ldots (1)$$

Differentiating w.r.t. y partially we have,

$$\frac{\partial z}{\partial y} = f(x)\, e^{y} - g(x)\, e^{-y} \qquad \ldots (2)$$

By data, $z = 0$ and $\dfrac{\partial z}{\partial y} = \sin x$ when $y = 0$.

Hence (1) & (2) becomes

$$0 = f(x) + g(x) \quad \text{and} \quad \sin x = f(x) - g(x)$$

By solving these simultaneously we get,

$$f(x) = \frac{\sin x}{2} \quad \text{and} \quad g(x) = -\frac{\sin x}{2}$$

We substitute these in (1).

Now, $z = \dfrac{\sin x}{2}\, e^{y} - \dfrac{\sin x}{2}\, e^{-y} = \sin x \cdot \dfrac{1}{2}\,(e^{y} - e^{-y}) = \sin x \sinh y$

Thus $\mathbf{z = \sin x \sinh y}$ is the required solution.

---

37. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + 3\dfrac{\partial z}{\partial x} - 4z = 0$ *subject to the conditions that* $z = 1$ *and* $\dfrac{\partial z}{\partial x} = y$ *when* $x = 0$.

>> Let us suppose that z is a function of x only. The given PDE assumes the form of an ODE

$$(D^2 + 3D - 4)\, z = 0 \quad \text{where } D = \frac{d}{dx}$$

A.E is $m^2 + 3m - 4 = 0$ or $(m-1)(m+4) = 0$

$\therefore \quad m = 1,\ m = -4.$

The solution of the ODE is given by

$$z = c_1 e^{x} + c_2 e^{-4x}$$

Solution of the PDE is given by

$$z = f(y)\, e^{x} + g(y)\, e^{-4x} \qquad \ldots (1)$$

Differentiating partially w.r.t. x we get,

$$\frac{\partial z}{\partial x} = f(y) \cdot e^{x} + g(y) \cdot e^{-4x} \cdot (-4) \qquad \ldots (2)$$

By data, $z = 1$ and $\dfrac{\partial z}{\partial x} = y$ when $x = 0$

Hence (1) and (2) becomes

$$1 = f(y) + g(y) \quad \text{and} \quad y = f(y) - 4g(y)$$

By solving these simultaneously we get,

$$f(y) = \frac{1}{5}(4+y) \text{ and } g(y) = \frac{1}{5}(1-y)$$

We substitute these in (1).

Thus $\quad \boldsymbol{z = \frac{1}{5}(4+y)\, e^{x} + \frac{1}{5}(1-y)\, e^{-4x}}$ is the required solution.

---

38. *Solve* $\dfrac{\partial^3 z}{\partial x^3} + 4\dfrac{\partial z}{\partial x} = 0$, *given that* $z = 0$, $\dfrac{\partial z}{\partial x} = 0$, $\dfrac{\partial^2 z}{\partial x^2} = 4$ *when* $x = 0$.

>> Let us suppose that z is a function of x only.

The given PDE assumes the form of an ODE,

$$(D^3 + 4D)\, z = 0 \text{ where } D = \frac{d}{dx}$$

A.E. is $m^3 + 4m = 0$ or $m(m^2 + 4) = 0 \quad \therefore \; m = 0, \; m = \pm 2i$

The solution of the ODE is given by

$$z = c_1 + c_2 \cos 2x + c_3 \sin 2x$$

Solution of the PDE is given by

$$z = f(y) + g(y) \cos 2x + h(y) \sin 2x \qquad \ldots (1)$$

Differentiating partially w.r.t. x twice, we have,

$$\frac{\partial z}{\partial x} = -g(y) \cdot 2 \sin 2x + h(y) \cdot 2 \cos 2x \qquad \ldots (2)$$

$$\frac{\partial^2 z}{\partial x^2} = -g(y) \cdot 4 \cos 2x - h(y) \cdot 4 \sin 2x \qquad \ldots (3)$$

By data, $z = 0,\ \frac{\partial z}{\partial x} = 0$ and $\frac{\partial^2 z}{\partial x^2} = 4,$ when $x = 0$

Hence (1), (2) and (3) becomes,

$$0 = f(y) + g(y),\quad 0 = 2h(y) \text{ and } 4 = -4g(y)$$

From these, we get $h(y) = 0,\ g(y) = -1$ and $f(y) = 1$

We substitute these in (1).

Thus $\mathbf{z = 1 - \cos 2x = 2\sin^2 x}$ is the required solution.

---

39. *Solve* $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$, *using the substitution* $\frac{\partial u}{\partial x} = v$.

>> The given PDE can be written as

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial u}{\partial x} \text{ since } \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

It can be further written as $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial x}$

Using $\frac{\partial u}{\partial x} = v$, we have, $\frac{\partial}{\partial y}(v) = v$ or $\frac{\partial v}{\partial y} - v = 0$

Let us suppose that v is a function of y only. Then we have

$$\frac{dv}{dy} - v = 0 \text{ or } (D-1)v = 0, \text{ where } D = \frac{d}{dy}$$

A.E. is $m - 1 = 0$ or $m = 1$.

The solution of the ODE is given by $v = c\,e^y$

Solution of the PDE is given by

$$v = f(x)\,e^y \text{ where } v = \frac{\partial u}{\partial x}$$

[*We can also obtain v by separation of variables*]

Now we have $\frac{\partial u}{\partial x} = f(x)\,e^y$

Integrating w.r.t. x treating y as constant we have,

$u = e^y \int f(x)\,dx + g(y)$ and let $\int f(x)\,dx = F(x)$.

Thus $\mathbf{u = F(x)\,e^y + g(y)}$ is the required solution.

---

40. *Solve* $\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial u}{\partial y} = 0$ *using the substitution* $\frac{\partial u}{\partial y} = v$

>> The given equation can be written as $\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial u}{\partial y} = 0$

Using $\frac{\partial u}{\partial y} = v$, the equation becomes $\frac{\partial^2 v}{\partial x^2} - v = 0$

Let us suppose that v is a function of x only. Then we have,

$$\frac{d^2 v}{dx^2} - v = 0 \text{ or } (D^2 - 1)\, v = 0, \text{ where } D = \frac{d}{dx}$$

A.E. is $m^2 - 1 = 0 \quad \therefore \; m = \pm 1$

The solution of the ODE is given by

$$v = c_1 e^x + c_2 e^{-x}$$

Solution of the PDE is given by,

$$v = f(y) \cdot e^x + g(y) \cdot e^{-x} \text{ where } v = \frac{\partial u}{\partial y}$$

Now we have, $\frac{\partial u}{\partial y} = f(y)\, e^x + g(y)\, e^{-x}$

Integrating w.r.t. y treating x as constant,

$$u = e^x \int f(y)\, dy + e^{-x} \int g(y)\, dy + H(x)$$

Let $\int f(y)\, dy = F(y)$ and $\int g(y)\, dy = G(y)$

Thus $\boldsymbol{u = F(y)\, e^x + G(y)\, e^{-x} + H(x)}$

EXERCISES

1. Solve : $\frac{\partial^3 z}{\partial x^2 \partial y} + 18\, xy^2 + \sin(2x - y) = 0$

2. Solve : $\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$, given that $\frac{\partial z}{\partial y} = -2\cos y$ when $x = 0$ and $z = 0$ when $y = n\pi$

3. Solve $\frac{\partial^2 z}{\partial x \, \partial t} = e^{-2t} \cos 3x$ subject to the conditions,

(1) $z(x, 0) = 0$ (2) $\frac{\partial z}{\partial t}(0, t) = 0$

4. Solve the system of equations, $\frac{\partial u}{\partial x} = \sin y + z$, $\frac{\partial u}{\partial y} = x \cos y - z$, $\frac{\partial u}{\partial z} = x - y$

5. Obtain the solution of $2 \frac{\partial^2 z}{\partial x^2} - 5 \frac{\partial^2 z}{\partial x \, \partial y} + 6 \frac{\partial^2 z}{\partial y^2} = 0$ by using the transformation, $v = 2x + y$ and $w = 3x + y$

6. Solve $\frac{\partial^2 z}{\partial y^2} + z = 0$ given that $z = \cos x$ and $\frac{\partial z}{\partial y} = \sin x$ when $y = 0$.

7. Solve $\frac{\partial^2 z}{\partial x^2} + 4z = 0$ given that when $x = 0$, $z = e^{2y}$ and $\frac{\partial z}{\partial x} = 2$.

8. Solve $\frac{\partial^2 z}{\partial x^2} - 16z = 0$ given that $z = 0$, $\frac{\partial z}{\partial x} = 4 \sin y$ when $x = 0$.

9. Solve $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} + 2z = 0$ given that $z = e^y$ and $\frac{\partial z}{\partial x} = 1$ when $x = 0$.

10. Solve $\frac{\partial^2 u}{\partial x \, \partial y} + 4 \frac{\partial u}{\partial y} = 0$ using the substitution $\frac{\partial u}{\partial y} = v$.

ANSWERS

1. $z = -x^3 y^3 + \frac{1}{4} \cos (2x - y) + x F(y) + G(y) + H(x)$

2. $z = -\sin y (\cos x + 1)$

3. $z = \frac{1}{6} \sin 3x (1 - e^{-2t})$

4. $u = x \sin y + xz - yz$

5. $z = f(2x + y) + g(3x + y)$

6. $z = \cos(x - y)$

7. $z = e^{2y} \cos 2x + \sin 2x$

8. $z = \sinh 4x \sin y$

9. $z = e^x \left[e^y \cos x + (1 - e^y) \sin x \right]$

10. $z = F(y) e^{-4x} + G(x)$

4.7 Solution of the Lagrange's linear PDE

Given a relation of the form $\phi(u, v) = 0$ where u and v are functions of x, y, z where $z = z(x, y)$, we have already discussed the method of forming a PDE by eliminating the arbitrary function ϕ (*article* 4.3. *Problems* 13 to 16). Now let us see the process involved in the formation of PDE from the relation $\phi(u, v) = 0$.

Differentiating partially w.r.t x and y by applying chain rule we have,

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] = 0$$

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] = 0$$

Transposing the second term in these equations onto the R.H.S and dividing one by the other we have,

$$\frac{\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z}p}{\dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}q} = \frac{\dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z}p}{\dfrac{\partial v}{\partial y} + \dfrac{\partial v}{\partial z}q}$$

i.e., $$\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right]\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right] - \left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right]\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right] = 0$$

i.e., $$p\left[\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial u}{\partial y}\right] + q\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right] = \left[\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\right]$$

This is of the form $Pp + Qq = R$, ... (1)

where, $P = \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial z}\dfrac{\partial u}{\partial y}$, $Q = \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z} - \dfrac{\partial v}{\partial x}\dfrac{\partial u}{\partial z}$, $R = \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x} - \dfrac{\partial v}{\partial y}\dfrac{\partial u}{\partial x}$

Equation (1) is a PDE of first order and first degree known as **Lagrange's linear equation** which has a solution of the form $\phi(u, v) = 0$

We now proceed to discuss the **method of solving Lagrange's linear PDE of the form $Pp + Qq = R$**

Let us consider two equations

$$u(x, y, z) = c_1, \quad v(x, y, z) = c_2$$

where c_1 and c_2 are constants. Taking differentials (*total derivative*) we have

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0$$

By the rule of cross multiplication we have,

$$\frac{dx}{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z}-\dfrac{\partial v}{\partial y}\dfrac{\partial u}{\partial z}}=\frac{dy}{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x}-\dfrac{\partial v}{\partial z}\dfrac{\partial u}{\partial x}}=\frac{dz}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}-\dfrac{\partial v}{\partial x}\dfrac{\partial u}{\partial y}}$$

i.e., $$\frac{dx}{-P}=\frac{dy}{-Q}=\frac{dz}{-R} \quad \text{or} \quad \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \ldots(2)$$

Equation (2) can be regarded as a system of equations (*simultaneous equations*) in three variables and relations $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ satisfy these equations. Thus $\phi(u,v)=0$ is a general solution of Lagrange's linear PDE.

Working procedure for problems

- Given the PDE in the form $Pp+Qq=R$ we form equation of the form $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ known as the *Auxilary Equation*. This system of equations can be solved as follows.

  - We can consider suitable pairs which can be put in forms like $f(x)\,dx=g(y)\,dy$, $g(y)\,dy=h(z)\,dz$, $f(x)\,dx=h(z)\,dz$ (*Separation of variables*) so that by integration we can get the relations in (x,y); (y,z); (z,x) as the case may be.

  or

  - We have a property in ratio and proportion that a ratio $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$ is also equal to $\frac{k_1a_1+k_2a_2+k_3a_3}{k_1b_1+k_2b_2+k_3b_3}$
  With reference to the Auxilary Equation we try to find multipliers $k_1, k_2, k_3;\ k_1', k_2', k_3'$ such that

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{k_1\,dx+k_2\,dy+k_3\,dz}{k_1P+k_2Q+k_3R}=\frac{k_1'\,dx+k_2'\,dy+k_3'\,dz}{k_1'P+k_2'Q+k_3'R}$$

- Integrating the two new expressions we obtain two relations connecting $x,\ y,\ z$.

- Suppose $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ are the two relations so obtained, then $\phi(u,v)=0$ constitutes a general solution of the PDE $Pp+Qq=R$

WORKED PROBLEMS

41. *Solve* : $x\,p + y\,q = z$

>> The given equation is of the form $P\,p + Q\,q = R$.

The auxilary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Consider $\frac{dx}{x} = \frac{dy}{y}$ which on integration will give

$\log x = \log y + \log c_1$ (*constant of integration is taken as* $\log c_1$ *only for convenience*)

i.e., $\log\left(x/y\right) = \log c_1$ or $x/y = c_1$

Also consider $\frac{dy}{y} = \frac{dz}{z}$ and we can similarly get $y/z = c_2$.

Thus a general solution of the PDE is given by

$$\phi\,(x/y,\; y/z) = 0$$

Note : *Solution can also be* $\phi\,(y/z,\; z/x) = 0$; $\phi\,(z/x,\; x/y) = 0$ *etc.*

---

42. *Solve* : $p\cot x + q\cot y = \cot z$

>> The given equation is of the form $P\,p + Q\,q = R$.

The auxilary equations are

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z} \qquad \ldots (1)$$

Taking the first and second terms we have $\tan x\,dx = \tan y\,dy$ which on integration will give

$$\log(\sec x) = \log(\sec y) + \log c_1$$

or $\log\left(\frac{\sec x}{\sec y}\right) = \log c_1$ or $\frac{\sec x}{\sec y} = c_1$

Similarly taking the second and third terms in (1) we obtain $\frac{\sec y}{\sec z} = c_2$

Thus a general solution of the PDE is given by

$$\phi\,(\sec x/\sec y,\; \sec y/\sec z) = 0$$

---

43. *Solve* : $y^2 z p = x^2 (z q + y)$

>> The given equation $y^2 z p - x^2 z q = x^2 y$ is of the form $P p + Q q = R$.
The auxilary equations are

$$\frac{dx}{y^2 z} = \frac{dy}{-x^2 z} = \frac{dz}{x^2 y} \qquad \ldots (1)$$

Taking the first and second terms we have

$$x^2 dx + y^2 dy = 0.$$

Integrating we get $x^3/3 + y^3/3 = c_1$ or $\mathbf{x^3 + y^3 = 3c_1}$

Also taking the last two terms in (1) we have $y\,dy + z\,dz = 0$. Integrating we get

$$\frac{y^2}{2} + \frac{z^2}{2} = c_2 \text{ or } \mathbf{y^2 + z^2 = 2c_2}$$

Thus a general solution of the PDE is given by

$$\boldsymbol{\phi(x^3 + y^3,\ y^2 + z^2) = 0}$$

---

44. *Solve* : $(y - z) p + (z - x) q = (x - y)$

>> The given equation is of the form $P p + Q q = R$.
The auxilary equations are

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y} \qquad \ldots (1)$$

Using multipliers 1, 1, 1 each ratio is equal to

$$\frac{dx + dy + dz}{y - z + z - x + x - y} = \frac{dx + dy + dz}{0}$$

$\therefore$ $dx + dy + dz = 0$ which on integration gives $\mathbf{x + y + z = c_1}$

Again by using multipliers x, y, z each ratio in (1) is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x y - xz + yz - x y + xz - yz} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$ which on integration gives

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2 \text{ or } \mathbf{x^2 + y^2 + z^2 = 2c_2}$$

Thus a general solution of the PDE is given by

$$\boldsymbol{\phi(x + y + z,\ x^2 + y^2 + z^2) = 0}$$

---

45. *Solve* : $(y^2 + z^2) p + x (y q - z) = 0$

>> The given equation $(y^2 + z^2) p + x y q = x z$ is of the form $P p + Q q = R$.

The auxilary equations are

$$\frac{dx}{y^2+z^2} = \frac{dy}{xy} = \frac{dz}{xz} \quad \ldots (1)$$

Taking the second and third terms we have,

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating we get $\log y = \log z + \log c_1$

i.e., $\log(y/z) = \log c_1$ or $y/z = c_1$

Using the multipliers $x, -y, -z$ each ratio in (1) is equal to

$$\frac{x\,dx - y\,dy - z\,dz}{xy^2 + xz^2 - xy^2 - xz^2} = \frac{x\,dx - y\,dy - z\,dz}{0}$$

$\therefore \quad x\,dx - y\,dy - z\,dz = 0.$

Integrating we get $\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = c_2$ or $x^2 - y^2 - z^2 = 2c_2$

Thus a general solution of the PDE is given by

$$\phi\,(y/z,\ x^2 - y^2 - z^2) = 0$$

---

46. Solve : $x(y^2 - z^2)\,p + y(z^2 - x^2)\,q = z(x^2 - y^2)$

\>> The given equation is of the form $Pp + Qq = R$. The auxilary equations are

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \quad \ldots (1)$$

Using the multipliers $1/x,\ 1/y,\ 1/z$ each ratio is equal to

$$\frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2} = \frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{0}$$

$\therefore \quad \frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$

Integrating we get

$$\log x + \log y + \log z = \log c_1$$

i.e., $\log(xyz) = \log(c_1)$ or $xyz = c_1$

Again using the multipliers x, y, z each ratio in (1) is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x^2y^2 - x^2z^2 + y^2z^2 - x^2y^2 + z^2x^2 - z^2y^2} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore \quad x\,dx + y\,dy + z\,dz = 0$

Integrating we get $x^2/2 + y^2/2 + z^2/2 = c_2$ or $x^2 + y^2 + z^2 = 2\,c_2$

Thus a general solution of the PDE is given by

$$\phi\,(xyz,\; x^2 + y^2 + z^2) = 0$$

---

47. *Solve* : $x\,(y^2 + z)\,p - y\,(x^2 + z)\,q = z\,(x^2 - y^2)$

>> The given equation is of the form $P\,p + Q\,q = R$. The auxilary equations are

$$\frac{dx}{x\,(y^2 + z)} = \frac{dy}{-y\,(x^2 + z)} = \frac{dz}{z\,(x^2 - y^2)} \qquad \ldots (1)$$

Taking the multipliers $1/x,\; 1/y,\; 1/z$ each ratio is equal to

$$\frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{0}$$

$\therefore \quad xyz = c_1$ *(Refer to the previous problem)*

Again taking multipliers $x,\; y,\; -1$ each ratio in (1) is equal to

$$\frac{x\,dx + y\,dy - dz}{x^2 y^2 + x^2 z - x^2 y^2 - y^2 z - x^2 z + y^2 z} = \frac{x\,dx + y\,dy - dz}{0}$$

$\therefore \quad x\,dx + y\,dy - dz = 0$

Integrating we get $x^2/2 + y^2/2 - z = c_2$ or $x^2 + y^2 - 2\,z = 2c_2$

Thus a general solution of the PDE is given by

$$\phi\,(xyz,\; x^2 + y^2 - 2\,z) = 0$$

---

48. *Find the general solution of* $xz\dfrac{\partial z}{\partial x} + yz\dfrac{\partial z}{\partial y} = xy$

>> The equation is of the form $Pp + Qq = R$ where $P = xz,\; Q = yz,\; R = xy$
The auxilary equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \qquad \ldots (1)$$

Taking the first and second terms we have, $\dfrac{dx}{x} = \dfrac{dy}{y}$ which on integration gives

$$\log x = \log y + \log c_1 \quad \text{or} \quad \log(x/y) = \log c_1 \quad \text{or} \quad x/y = c_1$$

Next, by using the multipliers $y, x, -2z$ each ratio in (1) is equal to

$$\frac{y\,dx + x\,dy - 2z\,dz}{xyz + xyz - 2xyz} = \frac{y\,dx + x\,dy - 2z\,dz}{0} = \frac{d(xy) - 2z\,dz}{0}$$

$\therefore\ d(xy) - d(z^2) = 0$, which on integration gives $xy - z^2 = c_2$

Thus a general solution of the PDE is given by

$$\phi\,(x/y,\ xy - z^2) = 0$$

---

49. *Solve :* $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

>> The given equation is of the form $Pp + Qq = R$. The auxilary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \qquad \ldots (1)$$

Using the multipliers $1/x^2,\ 1/y^2,\ 1/z^2$ each ratio is equal to

$$\frac{\frac{1}{x^2}\,dx + \frac{1}{y^2}\,dy + \frac{1}{z^2}\,dz}{(y-z)+(z-x)+(x-y)} = \frac{\frac{1}{x^2}\,dx + \frac{1}{y^2}\,dy + \frac{1}{z^2}\,dz}{0}$$

$\therefore\quad \frac{1}{x^2}\,dx + \frac{1}{y^2}\,dy + \frac{1}{z^2}\,dz = 0$, which on integration gives

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c_1 \quad \text{or} \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -c_1 = k_1$$

Again using the multipliers $1/x,\ 1/y,\ 1/z$ each ratio in (1) is equal to

$$\frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{x(y-z) + y(z-x) + z(x-y)} = \frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{0}$$

$\therefore\quad \frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$, which on integration gives

$$\log x + \log y + \log z = c_2 \quad \text{or} \quad \log(xyz) = \log k_2 \quad \text{or} \quad xyz = k_2$$

Thus a general solutions of the PDE is given by

$$\phi\,(1/x + 1/y + 1/z,\ xyz) = 0$$

---

50. *Solve :* $(y+z)p+(z+x)q = x+y$

>> The given equation is of the form $Pp+Qq = R$. The auxilary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \qquad \ldots(1)$$

Each ratio is equal to

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y-x} = \frac{dx-dz}{z-x} \qquad \ldots(2)$$

Taking the first two terms on integration we have,

$$\frac{1}{2}\log(x+y+z) = -\log(y-x)+\log c_1$$

i.e., $\log\left[\sqrt{x+y+z}\cdot(y-x)\right] = \log c_1$ or $\sqrt{x+y+z}\,(y-x) = c_1$

Again taking the last two terms in (2) we have after integration

$$-\log(y-x) = -\log(z-x)-\log c_2$$

i.e. $\log\left[\frac{y-x}{z-x}\right] = \log c_2$ or $\frac{y-x}{z-x} = c_2$

Thus a general solution of the PDE is given by

$$\phi\left(\sqrt{x+y+z}\,(y-x),\ \frac{y-x}{z-x}\right) = 0$$

---

51. *Solve :* $x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = (x-y)z$

>> The given equation $x^2p-y^2q = (x-y)z$ is of the form $Pp+Qq = R$.
The auxilary equations are

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{(x-y)z} \qquad \ldots(1)$$

From the first two relations we have on integration

$$\frac{-1}{x} = \frac{1}{y}+c_1 \quad \text{or} \quad \frac{1}{y}+\frac{1}{x} = -c_1$$

We also have from (1)

$$\frac{dx+dy}{x^2-y^2} = \frac{dz}{(x-y)z} \quad \text{or} \quad \frac{dx+dy}{x+y} = \frac{dz}{z}$$

Integrating we get $\log(x+y) = \log z+\log c_2$

i.e., $\log\left(\frac{x+y}{z}\right) = \log c_2$ or $\frac{x+y}{z} = c_2$

Thus a general solution of the PDE is given by

$$\phi\left(\frac{1}{y}+\frac{1}{x}, \frac{x+y}{z}\right) = 0$$

---

52. *Solve* : $y^2 p - x y q = x(z-2y)$

>> The given equation is of the form $Pp + Qq = R$. The auxilary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Consider $\frac{dx}{y^2} = \frac{dy}{-xy}$

i.e., $\frac{dx}{y} = \frac{dy}{-x}$ or $x\,dx + y\,dy = 0$

Integrating we get, $x^2/2 + y^2/2 = c_1$ or $x^2+y^2 = 2c_1$

Now consider $\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

ie., $\frac{dy}{-y} = \frac{dz}{z-2y}$

$$(z-2y)\,dy + y\,dz = 0$$

i.e., $(z\,dy + y\,dz) - 2y\,dy = 0$

ie., $d(yz) - 2y\,dy = 0.$

Integrating we get $yz - y^2 = c_2$

Thus a general solution of the PDE is given by

$$\phi(x^2+y^2,\ yz-y^2) = 0$$

---

53. *Solve* : $(x^2-yz)p + (y^2-zx)q = (z^2-xy)$

>> The given equation is of the form $Pp + Qq = R$.

The auxilary equations are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

Equivalently we can write in the form,

8

$$\frac{dx-dy}{(x^2-y^2)+z(x-y)} = \frac{dy-dz}{(y^2-z^2)+x(y-z)} = \frac{dz-dx}{(z^2-x^2)+y(z-x)}$$

ie., $$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

or $$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x} \qquad \ldots (1)$$

From the first and second terms of (1) we have,

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

$\Rightarrow$ $\log(x-y) = \log(y-z) + \log c_1$, on integration.

or $$\log\left(\frac{x-y}{y-z}\right) = \log c_1 \Rightarrow \frac{x-y}{y-z} = c_1$$

Similarly from the second and third terms of (1) we obtain $\dfrac{y-z}{z-x} = c_2$

Thus a general solution of the given equation is

$$\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$$

---

54. *Solve* : $(x^2-y^2-z^2)p + 2xyq = 2xz$

>> The given equation is of the form $Pp+Qq = R$.
The auxilary equations are

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \qquad \ldots (i)$$

Taking the second and third terms we have,

$$\frac{dy}{2xy} = \frac{dz}{2xz} \quad \text{or} \quad \frac{dy}{y} = \frac{dz}{z}$$

Integrating we get, $\log y = \log z + \log c_1$

ie., $\log(y/z) = \log c_1$ or $y/z = c_1$

Using multipliers x, y, z each ratio in (1) is equal to

$$\frac{x\,dx+y\,dy+z\,dz}{x^3-xy^2-xz^2+2xy^2+2xz^2} = \frac{x\,dx+y\,dy+z\,dz}{x^3+xy^2+xz^2} = \frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$$

Let us consider

$$\frac{dy}{2xy} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)} \quad \text{or} \quad \frac{dy}{y} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

Integrating we get, $\log y = \log(x^2+y^2+z^2) + \log c_2$

or $\quad \log\left(\frac{y}{x^2+y^2+z^2}\right) = \log c_2 \quad \Rightarrow \quad \frac{y}{x^2+y^2+z^2} = c_2$

Thus a general solution of the PDE is given by

$$\phi\left(y/z,\ y/x^2+y^2+z^2\right) = 0$$

---

55. *Solve :* $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} + (mx - ly) = 0$

>> The given equation is $(mz - ny)p + (nx - lz)q = (ly - mx)$

This is of the form $Pp + Qq = R$.

The auxilary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \qquad \ldots(1)$$

Using the multiplies $l,\ m,\ n$ each ratio is (1) is equal to

$$\frac{l\,dx + m\,dy + n\,dx}{lmz - nly + mnx - lmz + nly - mnx} = \frac{ldx + mdy + ndz}{0}$$

$\therefore \quad l\,dx + m\,dy + n\,dz = 0$

Integrating we get $\boldsymbol{lx + my + nz = c_1}$

Again using the multipliers $x,\ y,\ z$ each ratio in (1) is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{mxz - nxy + nxy - lyz + lyz - mxz} = \frac{xdx + ydy + zdz}{0}$$

$\therefore \quad x\,dx + y\,dy + z\,dz = 0$

Integrating we get, $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$ or $\boldsymbol{x^2 + y^2 + z^2 = 2c_2}$

Thus a general solution of the PDE is given by

$$\phi\left(lx + my + nz,\ x^2 + y^2 + z^2\right) = 0$$

EXERCISES

Solve :

1. $p \tan x + q \tan y = \tan z$
2. $yzp + zxq = xy$
3. $x(y-z)p + y(z-x)q = z(x-y)$
4. $(mz - ny)p + (nx - lz)q = ly - mx$
5. $\frac{y^2 z}{x} \cdot p + xzq = y^2$
6. $(y+z)p - (z+x)q = (x-y)$
7. $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$
8. $(y+zx)p - (x+yz)q = x^2-y^2$
9. $(y-zx)p + (yz+x)q = x^2+y^2$
10. $(y^2+z^2)p - xyq + zx = 0$

ANSWERS

1. $\phi(\sin y/\sin x,\ \sin z/\sin x) = 0$
2. $\phi(x^2-y^2,\ y^2-z^2) = 0$
3. $\phi(x+y+z,\ xyz) = 0$
4. $\phi(x^2+y^2+z^2,\ lx+my+nz) = 0$
5. $\phi(x^3-y^3,\ x^2-z^2) = 0$
6. $\phi(x+y+z,\ x^2+y^2-z^2) = 0$
7. $\phi(x^2+y^2-2z,\ xyz) = 0$
8. $\phi(x^2+y^2+z^2,\ xy+z) = 0$
9. $\phi(x^2-y^2+z^2,\ xy-z) = 0$
10. $\phi(y/z,\ x^2+y^2+z^2) = 0$

4.8 Solution of PDE by the method of separation of variables (Product method)

This method is applicable for solving a linear homogeneous PDE involving derivatives with respect to two independent variables. Solution of the PDE is determined through the solution of two ODEs.

The method is illustrated stepwise in respect of a PDE involving two indepedent variables x, y and $u = u(x, y)$

Working procedure for problems

- We assume the solution of the PDE in the form of a product.

That is $u = XY$ where $X = X(x)$ and $Y = Y(y)$

- $u = XY$ is substituted into the given PDE wherein the partial derivatives present in the equation converts into ordinary derivatives.

[*For example* $\frac{\partial u}{\partial x} = \frac{\partial (XY)}{\partial x} = Y\frac{dX}{dx}$ *since* $X = X(x)$ *and* $Y = Y(y)$]

- The resulting equation involving ordinary derivatives is rearranged in such a way that L.H.S is a function of x and R.H.S is a function of y (*or vice-versa*)

- We equate each side comprising ODEs to a common constant k.

  [*It is obvious that if* $f(x) = g(y)$ *then they must be equal to a constant.*]

- We solve the ODEs to obtain $X = X(x)$ and $Y = Y(y)$
- Substitution of X and Y into $u = XY$ results in the required solution of the PDE.

Remark : *If the given linear homogeneous PDE is of first order, we obviously get first order ODEs and they get solved by the known method : Separation of Variables. However if the linear homogeneous PDE is of order* ≥ 2 *we have to employ the known procedure of solving a linear homogeneous higher order ODE.*

WORKED PROBLEMS

56. *Solve* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 2(x+y)u$, *by the method of separation of variables.*

>> Let $u = XY$, where $X = X(x)$ & $Y = Y(y)$ be the solution of the given PDE.

Substituting into the given PDE we have, $\dfrac{\partial(XY)}{\partial x} + \dfrac{\partial(XY)}{\partial y} = 2(x+y)\,XY$

ie., $\quad Y\dfrac{dX}{dx} + X\dfrac{dY}{dy} = 2(x+y)\,XY.$

Dividing by XY we have, $\dfrac{1}{X}\dfrac{dX}{dx} + \dfrac{1}{Y}\dfrac{dY}{dy} = 2(x+y)$

or $\quad \dfrac{1}{X}\dfrac{dX}{dx} - 2x = -\dfrac{1}{Y}\dfrac{dY}{dy} + 2y$

Equating both sides to a common constant k we have,

$$\frac{1}{X}\frac{dX}{dx} - 2x = k \quad ; \quad -\frac{1}{Y}\frac{dY}{dy} + 2y = k$$

or $$\frac{1}{X}\frac{dX}{dx} = 2x + k \quad ; \quad \frac{1}{Y}\frac{dY}{dy} = 2y - k$$

$\therefore$ $$\frac{dX}{X} = (2x+k)\,dx \quad ; \quad \frac{dY}{Y} = (2y-k)\,dy$$

$\Rightarrow$ $$\int\frac{dX}{X} = \int(2x+k)\,dx + c_1 \quad ; \quad \int\frac{dY}{Y} = \int(2y-k)\,dy + c_2$$

ie., $$\log_e X = x^2 + kx + c_1 \quad ; \quad \log_e Y = y^2 - ky + c_2$$

or $$X = e^{x^2 + kx + c_1} \quad ; \quad Y = e^{y^2 - ky + c_2}$$

Hence $u = XY = e^{c_1 + c_2}\, e^{x^2 + kx + y^2 - ky}$ and let $c = e^{c_1 + c_2}$

Thus $\boldsymbol{u = c\,e^{x^2 + y^2 + k(x-y)}}$ is the required solution.

---

57. *Solve* $x^2 \dfrac{\partial u}{\partial x} + y^2 \dfrac{\partial u}{\partial y} = 0$ *by the method of separation of variables.*

>> Let $u = XY$, where $X = X(x)$ & $Y = Y(y)$ be the solution of the given PDE. Substituting into the given PDE we have,

$$x^2 \frac{\partial}{\partial x}(XY) + y^2 \frac{\partial}{\partial y}(XY) = 0$$

i.e., $$x^2 Y \frac{dX}{dx} + y^2 X \frac{dY}{dy} = 0$$

Dividing by XY we have,

$$\frac{x^2}{X}\frac{dX}{dx} = -\frac{y^2}{Y}\frac{dY}{dy}$$

Equating both sides to a common constant k, we have

$$\frac{x^2}{X}\frac{dX}{dx} = k \quad ; \quad -\frac{y^2}{Y}\frac{dY}{dy} = k$$

$$\therefore \quad \frac{dX}{X} = \frac{k}{x^2}\,dx \quad ; \quad \frac{dY}{Y} = -\frac{k}{y^2}\,dy$$

$$\Rightarrow \quad \int \frac{dX}{X} = k\int \frac{1}{x^2}\,dx + c_1 \quad ; \quad \int \frac{dY}{Y} = k\int \frac{-1}{y^2}\,dy + c_2$$

ie., $$\log_e X = k \cdot \frac{-1}{x} + c_1 \quad ; \quad \log_e Y = k\,\frac{1}{y} + c_2$$

or $$X = e^{(-k/x) + c_1} \quad ; \quad Y = e^{(k/y) + c_2}$$

Hence $u = XY = e^{c_1 + c_2}\, e^{-k/x + k/y}$ and let $c = e^{c_1 + c_2}$

Thus $\;\; u = c\,e^{k(1/y - 1/x)}$ is the required solution.

---

58. *Solve by the method of separation of variables* $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$ *where* $u(x, 0) = 6e^{-3x}$

>> Let $u = XT$, where $X = X(x)$ & $T = T(t)$ be the solution of the given PDE.

Substituting into the given PDE, we have $T\dfrac{dX}{dx} = 2X\dfrac{dT}{dt} + XT$

Dividing by XT we get, $\dfrac{1}{X}\dfrac{dX}{dx} = \dfrac{2}{T}\dfrac{dT}{dt} + 1$

Equating both sides to a common constant k we have,

$$\frac{1}{X}\frac{dX}{dx} = k \quad ; \quad \frac{2}{T}\frac{dT}{dt} + 1 = k$$

ie., $$\frac{1}{X}\frac{dX}{dx} = k \quad ; \quad \frac{2}{T}\frac{dT}{dt} = k - 1$$

$\therefore$ $$\frac{1}{X}\, dX = k\, dx \quad ; \quad \frac{1}{T}\, dT = \frac{k-1}{2}\, dt$$

$\Rightarrow$ $$\int \frac{1}{X}\, dX = k \int dx + c_1 \; ; \quad \int \frac{1}{T}\, dT = \frac{k-1}{2} \int dt + c_2$$

ie., $$\log_e X = kx + c_1 \quad ; \quad \log_e T = \left(\frac{k-1}{2}\right) t + c_2$$

or $$X = e^{kx + c_1} \quad ; \quad T = e^{(k-1)\, t/2 + c_2}$$

Hence $u = XT = e^{c_1 + c_2}\, e^{kx + (k-1)t/2}$ and let $c = e^{c_1 + c_2}$

Thus $\boldsymbol{u = c\, e^{kx + (k-1)\, t/2}}$ **is the general solution.**

Further by data, $u(x, 0) = 6e^{-3x}$ That is $u = 6e^{-3x}$, when $t = 0$

Hence we have $6e^{-3x} = c\, e^{kx}$

Comparing we get $c = 6$ and $k = -3$.

Thus the required particular solution is given by $\boldsymbol{u = 6\, e^{-3x - 2t}}$

---

59. *Solve by the method of separation of variables* $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, *given that* $u(0, y) = 2e^{5y}$

>> Let $u = XY$, where $X = X(x)$ & $Y = Y(y)$ be the solution of the given PDE. Substituting into the given PDE we have,

$$4Y\frac{dX}{dx} + X\frac{dY}{dy} = 3XY$$

Dividing by XY we have,

$$\frac{4}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 3 \quad \text{or} \quad \frac{4}{X}\frac{dX}{dx} = 3 - \frac{1}{Y}\frac{dY}{dy}$$

Equating both sides to a common constant k we have,

$$\frac{4}{X}\frac{dX}{dx} = k \quad ; \quad 3 - \frac{1}{Y}\frac{dY}{dy} = k$$

ie., $\frac{4}{X}\frac{dX}{dx} = k$; $\frac{1}{Y}\frac{dY}{dy} = 3-k$

$\therefore$ $\frac{1}{X}\,dX = \frac{k}{4}\,dx$; $\frac{1}{Y}\,dY = (3-k)\,dy.$

$\Rightarrow$ $\int \frac{1}{X}\,dX = \frac{k}{4}\int dx + c_1$; $\int \frac{1}{Y}\,dY = (3-k)\int dy + c_2$

ie., $\log_e X = \frac{kx}{4} + c_1$; $\log_e Y = (3-k)\,y + c_2$

$X = e^{(kx/4)+c_1}$; $Y = e^{(3-k)y+c_2}$

Hence $u = XY = e^{c_1+c_2}\,e^{(kx/4)+(3-k)y}$ and let $c = e^{c_1+c_2}$

Thus $u = u(x, y) = c\,e^{(kx/4)+(3-k)y}$ is the general solution.

Further by data, $u(0, y) = 2\,e^{5y}$

The general solution becomes $2\,e^{5y} = c\,e^{(3-k)y}$

Comparing we have, $c = 2$ and $3-k = 5$ or $k = -2$

Thus the required particular solution is given by $u = 2\,e^{(-x/2)+5y}$

---

60. *Solve $u_{xy} = u$ by the method of separation of variables.*

>> Let $u = XY$, where $X = X(x)$ & $Y = Y(y)$ be the solution of the given PDE.

Then $u_{xy} = \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(XY)\right)$

ie., $= \frac{\partial}{\partial x}\left(X\frac{dY}{dy}\right) = \frac{dY}{dy}\cdot\frac{dX}{dx}$

Substituting into the given PDE we have $\frac{dY}{dy}\cdot\frac{dX}{dx} = XY$

Dividing by XY, we have $\left(\frac{1}{X}\frac{dX}{dx}\right)\cdot\left(\frac{1}{Y}\frac{dY}{dy}\right) = 1$

i.e., $\frac{1}{X}\frac{dX}{dx} = \frac{1}{\frac{1}{Y}\frac{dY}{dy}}$

Equating both sides to a common constant k we have,

$$\frac{1}{X}\frac{dX}{dx} = k, \quad ; \quad \frac{1}{\frac{1}{Y}\frac{dY}{dy}} = k \quad \text{or} \quad \frac{1}{Y}\frac{dY}{dy} = \frac{1}{k}$$

$$\therefore \quad \frac{1}{X}\,dX = k\,dx \quad ; \quad \frac{1}{Y}\,dY = \frac{1}{k}\,dy$$

$$\Rightarrow \quad \int \frac{1}{X}\,dX = k\int dx + c_1 \; ; \quad \int \frac{1}{Y}\,dY = \frac{1}{k}\int dy + c_2$$

$$ie., \quad \log_e X = kx + c_1 \quad ; \quad \log_e Y = \frac{y}{k} + c_2$$

$$\text{or} \quad X = e^{kx + c_1} \quad ; \quad Y = e^{(y/k) + c_2}$$

Hence $u = XY = e^{c_1 + c_2}\, e^{kx + (y/k)}$ and let $c = e^{c_1 + c_2}$

Thus $u = c\,e^{kx + (y/k)}$ is the required solution.

---

61. *Solve :* $p\,y^3 + q\,x^2 = 0$ *by the method of separation of variables.*

>> We have the PDE : $y^3 \dfrac{\partial z}{\partial x} + x^2 \dfrac{\partial z}{\partial y} = 0$

Let $z = XY$, where $X = X(x)$ and $Y = Y(y)$ be the solution of the given PDE. Substituting into the given PDE we have,

$$y^3\, Y \frac{dX}{dx} + x^2\, X \frac{dY}{dy} = 0$$

Dividing by $XY x^2 y^3$ we have

$$\frac{1}{x^2 X}\frac{dX}{dx} = \frac{-1}{y^3 Y}\frac{dY}{dy}$$

Equating both sides to a common constant k we have

$$\frac{1}{x^2 X}\frac{dX}{dx} = k \quad ; \quad \frac{-1}{y^3 Y}\frac{dY}{dy} = k$$

$$\therefore \quad \frac{1}{X}\,dX = k\,x^2\,dx \quad ; \quad \frac{1}{Y}\,dY = -k\,y^3\,dy$$

$$\Rightarrow \quad \int \frac{dX}{X} = \int k\,x^2\,dx + c_1 \quad ; \quad \int \frac{1}{Y}\,dY = -\int k\,y^3\,dy + c_2$$

$$i.e., \quad \log X = \frac{k\,x^3}{3} + c_1 \quad ; \quad \log Y = \frac{-k\,y^4}{4} + c_2$$

or $X = e^{kx^3/3 + c_1}$; $Y = e^{-ky^4/4 + c_2}$

Hence $z = XY = e^{c_1 + c_2}\, e^{kx^3/3 - ky^4/4}$ and let $c = e^{c_1 + c_2}$

Thus $z = c\,e^{k(x^3/3 - y^4/4)}$ is the required solution.

---

62. *Solve : $xp = yq$ by the method of separation of variables.*

>> We have the PDE : $x\dfrac{\partial z}{\partial x} = y\dfrac{\partial z}{\partial y}$

Let $z = XY$, where $X = X(x)$ and $Y = Y(y)$ be the solution of the given PDE.

Substituting into the given PDE we have,

$$xY\frac{dX}{dx} = yX\frac{dY}{dy}$$

Dividing by XY we have,

$$\frac{x}{X}\frac{dX}{dx} = \frac{y}{Y}\frac{dY}{dy}$$

Equating both sides to a common constant k we have,

$$\frac{x}{X}\frac{dX}{dx} = k \quad ; \quad \frac{y}{Y}\frac{dY}{dy} = k$$

$$\therefore \quad \frac{1}{X}\,dX = \frac{k}{x}\,dx \quad ; \quad \frac{1}{Y}\,dY = \frac{k}{y}\,dy$$

$$\Rightarrow \quad \int\frac{1}{X}\,dX = k\int\frac{dx}{x} + c_1 \quad ; \quad \int\frac{1}{Y}\,dY = k\int\frac{dy}{y} + c_2$$

i.e., $\log X = k\log x + c_1$; $\log Y = k\log y + c_2$

i.e., $\log X = \log x^k + \log c_1'$; $\log Y = \log y^k + \log c_2'$

i.e., $\log X = \log(c_1' x^k)$; $\log Y = \log(c_2' y^k)$

$\Rightarrow$ $X = c_1' x^k$; $Y = c_2' y^k$

Hence $z = XY = c_1' c_2' x^k y^k$ and let $c = c_1' c_2'$

Thus $z = c(xy)^k$ is the required solution.

---

63. *Solve by the method of separation of variables* $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$

>> Let $z = XY$, where $X = X(x)$ and $Y = Y(y)$ be the solution of the given PDE. Substituting into the given PDE we have

$$\frac{\partial^2}{\partial x^2}(XY) - 2\frac{\partial}{\partial x}(XY) + \frac{\partial}{\partial y}(XY) = 0$$

i.e., $$Y\frac{d^2 X}{dx^2} - 2Y\frac{dX}{dx} + X\frac{dY}{dy} = 0$$

Dividing by XY we have, $\dfrac{1}{X}\dfrac{d^2X}{dx^2} - 2\dfrac{1}{X}\dfrac{dX}{dx} = -\dfrac{1}{Y}\dfrac{dY}{dy}$

Equating both sides to a common constant k, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = k \quad\text{and}\quad -\frac{1}{Y}\frac{dY}{dy} = k$$

i.e., $$\frac{d^2X}{dx^2} - 2\frac{dX}{dx} - kX = 0 \quad\text{and}\quad \frac{dY}{dy} + kY = 0$$

Both are linear homogeneous ODE with constant coefficients.

A.Es are $m^2 - 2m - k = 0$ and $m + k = 0$

$\therefore$ roots of the AE are $m = \dfrac{-(-2) \pm \sqrt{4+4k}}{2}$ and $m = -k$

i.e., $m = 1 \pm \sqrt{1+k}$ and $m = -k$

$\therefore$ solution of ODEs are $X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$ and $Y = c_3 e^{-ky}$

i.e., $X = e^x\left(c_1 e^{(\sqrt{1+k})x} + c_2 e^{(-\sqrt{1+k})x}\right)$ and $Y = c_3 e^{-ky}$

Hence $z = XY = e^x\left(c_1 e^{(\sqrt{1+k})x} + c_2 e^{(-\sqrt{1+k})x}\right) \cdot c_3 e^{-ky}$

Let $c_1 c_3 = A$ and $c_2 c_3 = B$. Also we assume $k > -1$

Thus $\boldsymbol{u = e^{x-ky}\left(A\, e^{(\sqrt{1+k})x} + B\, e^{(-\sqrt{1+k})x}\right)}$ is the required solution.

Note : *Y can also be obtained by separating the variables like earlier problems.*

---

64. *Solve by the method of separation of variables,*

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

>> Let $u = RT$, where $R = R(r)$, and $T = T(\theta)$ be the solution of the given PDE.

Substituting into the given PDE we have,

$$r^2 \frac{\partial^2}{\partial r^2}(RT) + r\frac{\partial}{\partial r}(RT) + \frac{\partial^2}{\partial \theta^2}(RT) = 0$$

i.e., $$r^2 T \frac{d^2 R}{d r^2} + r T \frac{dR}{dr} + R \frac{d^2 T}{d\theta^2} = 0$$

Dividing by RT we have $\dfrac{r^2}{R}\dfrac{d^2 R}{d r^2} + \dfrac{r}{R}\dfrac{dR}{dr} = -\dfrac{1}{T}\dfrac{d^2 T}{d\theta^2}$

Equating both sides to a common constant k, we have

$$\frac{r^2}{R}\frac{d^2 R}{d r^2} + \frac{r}{R}\frac{dR}{dr} = k \text{ and } -\frac{1}{T}\frac{d^2 T}{d\theta^2} = k$$

i.e., $$r^2 \frac{d^2 R}{d r^2} + r\frac{dR}{dr} - kR = 0 \qquad \ldots(1)$$

and $$\frac{d^2 T}{d\theta^2} + kT = 0 \qquad \ldots(2)$$

(1) is in the form of Cauchy's homogeneous linear equation. The DE is solved by reducing into an ODE with constant coefficients using a substitution.

Put $\log r = t$ or $r = e^t$. Then we know that

$$r\frac{dR}{dr} = \frac{dR}{dt} \text{ and } r^2\frac{d^2 R}{d r^2} = \frac{d^2 R}{d t^2} - \frac{dR}{dt}$$

$\therefore$ (1) becomes $\left[\dfrac{d^2 R}{d t^2} - \dfrac{dR}{dt}\right] + \dfrac{dR}{dt} - kR = 0$

i.e., $$\frac{d^2 R}{d t^2} - kR = 0.$$

This is a linear homogeneous ODE with constant coefficients and the A.E is $m^2 - k = 0$. Therefore, $m = \pm\sqrt{k}\ (k > 0)$

Hence $R = c_1 e^{\sqrt{k}\,t} + c_2 e^{-\sqrt{k}\,t}$

i.e., $$R = c_1 (e^t)^{\sqrt{k}} + c_2 (e^t)^{-\sqrt{k}}$$

or $$R = c_1 r^{\sqrt{k}} + c_2 r^{-\sqrt{k}} \text{ since } r = e^t$$

Equation (2) is also a linear homogeneous ODE with constant coefficients.

A.E is $m^2 + k = 0 \quad \therefore\ m = \pm\sqrt{-k}$ or $m = \pm i\sqrt{k}$

Hence $T = c_3 \cos\sqrt{k}\,\theta + c_4 \sin\sqrt{k}\,\theta$

Thus the required solution is given by

$$u = RT = \left(c_1 r^{\sqrt{k}} + c_2 r^{-\sqrt{k}}\right)\left(c_3 \cos \sqrt{k}\,\theta + c_4 \sin \sqrt{k}\,\theta\right)$$

---

65. *Solve by the method of separation of variables the PDE* $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial y} + 2z$ *subject to the conditions* $z(0, y) = 0$ *and* $z_x(0, y) = e^{2y}$

>> Let $z = XY$, where $X = X(x)$ & $Y = Y(y)$ be the solution of the given PDE.

Substituting into the given PDE we have, $\dfrac{\partial^2}{\partial x^2}(XY) = \dfrac{\partial}{\partial y}(XY) + 2XY$

i.e., $Y \dfrac{d^2 X}{dx^2} = X \dfrac{dY}{dy} + 2XY$

Dividing by XY we have $\dfrac{1}{X}\dfrac{d^2 X}{dx^2} = \dfrac{1}{Y}\dfrac{dY}{dy} + 2$

Equating each side to a common constant k we have,

$$\frac{1}{X}\frac{d^2 X}{dx^2} = k \text{ and } \frac{1}{Y}\frac{dY}{dy} + 2 = k$$

i.e., $\dfrac{d^2 X}{dx^2} - kX = 0$ and $\dfrac{dY}{dy} - (k-2)Y = 0$

These are linear homogeneous ODE with constant coefficients.

A.Es are $m^2 - k = 0$ and $m - (k-2) = 0$

$\therefore \quad m = \pm\sqrt{k}$ and $m = (k-2)$ respectively, where $k > 0$

The solutions are $X = c_1 e^{\sqrt{k}\,x} + c_2 e^{-\sqrt{k}\,x}$ and $Y = c_3 e^{(k-2)y}$

Hence $z = XY = (c_1 e^{\sqrt{k}\,x} + c_2 e^{-\sqrt{k}\,x}) \cdot c_3 e^{(k-2)y}$

i.e., $z = (Ae^{\sqrt{k}\,x} + Be^{-\sqrt{k}\,x})e^{(k-2)y}$, where $A = c_1 c_3$ and $B = c_2 c_3$

Thus the general solution is given by

$$z = (A e^{\sqrt{k}\,x} + B e^{-\sqrt{k}\,x})\, e^{(k-2)y} \quad \ldots (1)$$

Differentiating (1) partially w.r.t. x we have,

$$z_x = \sqrt{k}\,(A e^{\sqrt{k}\,x} - B e^{-\sqrt{k}\,x})\, e^{(k-2)y} \quad \ldots (2)$$

The given conditions are $z = 0$ and $z_x = e^{2y}$ when $x = 0$.

Hence (1) and (2) becomes

$$(A+B)\ e^{(k-2)y} = 0 \qquad \dots (3)$$

$$\sqrt{k}\ (A-B)\ e^{(k-2)y} = e^{2y} \qquad \dots (4)$$

Since $e^{(k-2)y}$ cannot be zero we must have $A+B = 0$ or $B = -A$ and using this in (4) we get

$$2\sqrt{k}\ A\ e^{(k-2)y} = e^{2y}$$

$\Rightarrow$ $\quad k-2 = 2$ and $2\sqrt{k}\ A = 1$

$\therefore$ $\quad k = 4$ and $A = 1/4$. Also $B = -1/4$

Now (1) becomes

$$z = \frac{1}{4}\,(e^{2x} - e^{-2x})\,e^{2y} = \frac{1}{2}\sin h\,2x\,e^{2y}$$

Thus the required particular solution of the PDE is given by

$$z = (1/2)\ \sin h\,2x\ e^{2y}$$

EXERCISES

Solve the following PDE by the method of separation of variables.

1. $x\,\dfrac{\partial u}{\partial x} + y\,\dfrac{\partial u}{\partial y} = 0$ 2. $\dfrac{\partial u}{\partial x} - y\,\dfrac{\partial u}{\partial y} = 0$ 3. $2x\,\dfrac{\partial u}{\partial x} - 3y\,\dfrac{\partial u}{\partial y} = 0$

4. $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3\,(x^2+y^2)\,u$ 5. $\dfrac{\partial u}{\partial x} + 2\,\dfrac{\partial u}{\partial y} = u$ where $u = 6e^{-3x}$ when $y = 0$

6. $\dfrac{\partial^2 u}{\partial x^2} - 4\,\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$ 7. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ 8. $\dfrac{\partial^2 u}{\partial x^2} - 2\,\dfrac{\partial u}{\partial x} + \dfrac{\partial^2 u}{\partial y^2} = 0$

ANSWERS

1. $u = c\,\left(x/y\right)^k$ 2. $u = c\,e^{kx}\,y^k$ 3. $u = c\,x^{k/2}\,y^{k/3}$

4. $u = c\,e^{x^3+y^3+k(x-y)}$ 5. $u = 6\,e^{-3x+2y}$

6. $u = \left(Ae^{(2+\sqrt{4-k})x} + Be^{(2-\sqrt{4-k})x}\right)e^{-ky},\ \ k < 4$

7. $u = (c_1\,e^{\sqrt{k}x} + c_2\,e^{-\sqrt{k}x})\ (c_3\cos\sqrt{k}\,y + c_4\sin\sqrt{k}y),\ \ k > 0$

8. $u = [\,c_1\,e^{(1+\sqrt{1+k})x} + c_2\,e^{(1-\sqrt{1+k})x}\,]\,[\,c_3\cos\sqrt{k}y + c_4\sin\sqrt{k}y\,],\ \ k > 0$

PART B

Unit - V

Integral Calculus

5.1 Introduction

We are already conversant with the indefinite and definite integrals of a function of a single independent variable along with the applications.

In this chapter the concept is discussed for a function of two and three independent variables along with the applications.

Further we also discuss two special functions 'Beta function' and 'Gamma function' defined in the form of definite integrals.

5.2 Multiple integrals

In this topic we discuss a repeated process of integ:ation of a function of two and three variables referred to as *double integrals* : $\iint f(x, y)\, dx\, dy$ and *triple integrals* : $\iiint f(x, y, z)\, dx\, dy\, dz$.

The principle of partial differentiation is adopted here in the process of integration.

For example :

(i) $$\iint (x+y)\, dx\, dy = \int\left(\frac{x^2}{2} + y \cdot x\right) dy = \frac{x^2}{2} \cdot y + \frac{y^2}{2} \cdot x = \frac{xy}{2}(x+y)$$

(ii) $$\int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} xyz\; dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{2} xy\left[\frac{z^2}{2}\right]_0^3 dy\, dx = \int_{x=0}^{1} \int_{y=0}^{2} xy\left(\frac{9}{2} - 0\right) dy\, dx = \frac{9}{2} \int_{x=0}^{1} \int_{y=0}^{2} xy\, dy\, dx$$

$$= \frac{9}{2} \int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_0^2 dx = \frac{9}{2} \int_{x=0}^{1} x\left(\frac{2^2}{2} - 0\right) dx = 9\int_{x=0}^{1} x\, dx = 9\left[\frac{x^2}{2}\right]_0^1 = \frac{9}{2}$$

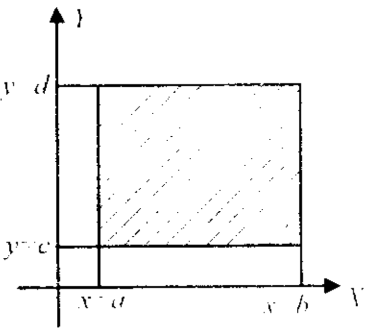
5.21 Geometrical meaning

$\int_{x=a}^{b} \int_{y=c}^{d} f(x, y)\, dy\, dx$ can be regarded as the integral over the region bounded by the rectangle with sides $x = a$, $x = b$, $y = c$, $y = d$.

The integral can also be evaluated by writing in the form

$\int_{y=c}^{d} \int_{x=a}^{b} f(x, y)\, dx\, dy$ and the value will be the same.

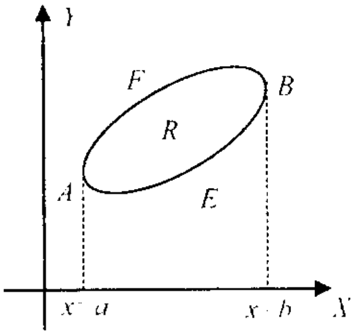
That is to say that when the limits are constant the integral can be evaluated in either way.



If R is a region of the $x-y$ plane bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and the lines $x = a$, $x = b$ we have

$$\iint_R f(x, y)\, dx\, dy = \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} f(x, y)\, dy\, dx$$

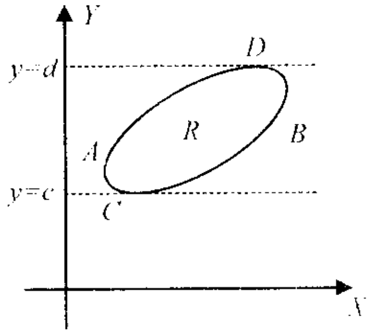
where $y = y_1(x)$ and $y = y_2(x)$ are the equations of the lower and upper part of the boundary curve respectively being AEB and AFB.



Further the integral can also be expressed in the form

$$\iint_R f(x, y)\, dx\, dy = \int_{y=c}^{d} \int_{x=x_1(y)}^{x_2(y)} f(x, y)\, dx\, dy$$

where $x = x_1(y)$, $x = x_2(y)$ are the equations of the left and right part of the boundary curve respectively being CAD and CBD.



It should be observed that if a function of x is involved as a limit in the double integral it corresponds to y in which case the limits for x will be constant. Similar argument holds good for a function of y in the limit and also in the case of triple integral involving three variables x, y, z

A form of triple integral is as follows.

$$I = \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} \int_{z=z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

WORKED PROBLEMS

1. *Evaluate* $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$

\>> We have $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy\, dy\, dx$

$$I = \int_{x=0}^{1} x \left[\frac{y^2}{2} \right]_x^{\sqrt{x}} dx = \int_{x=0}^{1} \frac{x}{2} \left[(\sqrt{x})^2 - x^2 \right] dx$$

$$= \frac{1}{2} \int_0^1 x (x - x^2)\, dx$$

$$= \frac{1}{2} \int_0^1 (x^2 - x^3)\, dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} \left[\left(\frac{1}{3} - \frac{1}{4} \right) - 0 \right] = \frac{1}{24}$$

Thus **$I = 1/24$**

---

2. *Evaluate* $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$

\>> We have $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$

$$= \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3} \right]_{y=x}^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} \left[x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3} \right] dx$$

$$I = \int_{x=0}^{1} \left[x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3} \right] dx$$

$$= \left[\frac{x^{7/2}}{7/2} + \frac{1}{3}\frac{x^{5/2}}{5/2} - \frac{4}{3}\cdot\frac{x^4}{4} \right]_{x=0}^{1}$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{9}{105} = \frac{3}{35}$$

Thus **$I = 3/35$**

---

3. *Evaluate :* $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \; dx \, dy$

>> We have $I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} x^3 y \; dx \, dy$

$$I = \int_{y=0}^{1} y \left[\frac{x^4}{4} \right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{4} \int_{y=0}^{1} y\,(1-y^2)^2 \, dy = \frac{1}{4} \int_{y=0}^{1} y\,(1 - 2y^2 + y^4)\, dy$$

i.e., $$= \frac{1}{4} \int_{y=0}^{1} (y - 2y^3 + y^5)\, dy$$

$$= \frac{1}{4} \left[\frac{y^2}{2} - \frac{y^4}{2} + \frac{y^6}{6} \right]_0^1 = \frac{1}{4} \left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right] = \frac{1}{24}$$

Thus **$I = 1/24$**

---

4. *Evaluate* $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$

>> $$I=\int_{x=-c}^{c}\int_{y=-b}^{b}\int_{z=-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$$

$$=\int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2z+y^2z+\frac{z^3}{3}\right]_{z=-a}^{a}dy\,dx$$

$$=\int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2(a+a)+y^2(a+a)+(a^3/3+a^3/3)\right]dy\,dx$$

$$=\int_{x=-c}^{c}\left[2a\,x^2y+2a(y^3/3)+(2a^3/3)(y)\right]_{y=-b}^{b}dx$$

$$=\int_{x=-c}^{c}[\,2a\,x^2(b+b)+2a(b^3/3+b^3/3)+(2a^3/3)(b+b)\,]\,dx$$

$$=\left[(4ab)(x^3/3)+(4ab^3/3)(x)+4a^3b/3\,(x)\right]_{-c}^{c}$$

$$=(4ab)(c^3/3+c^3/3)+(4ab^3/3)(c+c)+(4a^3b/3)(c+c)$$

$$=\frac{8abc^3}{3}+\frac{8ab^3c}{3}+\frac{8a^3bc}{3}=\frac{8abc(a^2+b^2+c^2)}{3}$$

Thus $\boldsymbol{I=8abc(a^2+b^2+c^2)/3}$

---

5. *Evaluate* $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

>> $$I=\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$$

$$=\int_{z=-1}^{1}\int_{x=0}^{z}\left[xy+\frac{y^2}{2}+zy\right]_{y=x-z}^{x+z}dx\,dz$$

$$I = \int_{z=-1}^{1} \int_{x=0}^{z} \left\{ x(\overline{x+z} - \overline{x-z}) + \frac{1}{2}\left[(x+z)^2 - (x-z)^2\right] + z(\overline{x+z} - \overline{x-z}) \right\} dx\,dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} (2xz + 2xz + 2z^2)\,dx\,dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} (4xz + 2z^2)\,dx\,dz$$

$$= \int_{z=-1}^{1} \left[z(2x^2) + 2z^2(x) \right]_{x=0}^{z} dz = \int_{z=-1}^{1} (2z^3 + 2z^3)\,dz$$

$$= \int_{-1}^{1} 4z^3\,dz = \left[z^4 \right]_{-1}^{1} = 0$$

Thus $I = 0$

---

6. *Evaluate* $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

>> $$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy(1 - x^2 - y^2)\,dy\,dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left[x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4} \right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \frac{1}{4}\left[2xy^2 - 2x^3y^2 - xy^4 \right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$I = \frac{1}{8}\int_{x=0}^{1} [\,2x(1-x^2) - 2x^3(1-x^2) - x(1-x^2)^2\,]\,dx$$

$$= \frac{1}{8}\int_{x=0}^{1} (2x - 2x^3 - 2x^3 + 2x^5 - x + 2x^3 - x^5)\,dx$$

$$= \frac{1}{8}\int_{x=0}^{1} (x^5 - 2x^3 + x)\,dx$$

$$= \frac{1}{8}\left[\frac{x^6}{6} - \frac{x^4}{2} + \frac{x^2}{2}\right]_{x=0}^{1}$$

$$= \frac{1}{8}\left[\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right] = \frac{1}{48}$$

Thus $I = 1/48$

---

7. *Evaluate* $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$

\>> $$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2 - z^2}}$$

We shall first integrate w.r.t z by treating x, y as constants and let $k = \sqrt{1-x^2-y^2}$ for convenience.

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{k} \frac{dz}{\sqrt{k^2 - z^2}}\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{k}\right]_{z=0}^{k} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} (\sin^{-1} 1 - \sin^{-1} 0)\,dy\,dx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left(\frac{\pi}{2} - 0\right) dy\, dx$$

$$= \frac{\pi}{2} \int_{x=0}^{1} [\, y\,]_0^{\sqrt{1-x^2}}\, dx = \frac{\pi}{2} \int_{x=0}^{1} \sqrt{1-x^2}\, dx$$

But $\displaystyle\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

Hence $\displaystyle I = \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_{x=0}^{1}$

$$= \frac{\pi}{2}\left[0 + \frac{1}{2}(\sin^{-1} 1 - \sin^{-1} 0)\right] = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}$$

Thus $\boldsymbol{I = \pi^2 / 8}$

---

8. *Evaluate* $\displaystyle\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

\>> We have $\displaystyle I = \int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y}\, e^z\, dz\, dy\, dx$

$$I = \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y}\left[\, e^z\, \right]_0^{x+y} dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y}\, [\, e^{x+y} - 1\,]\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} (\, e^{2x}\cdot e^{2y} - e^x \cdot e^y\,)\, dy\, dx$$

$$= \int_{x=0}^{a} \left\{ e^{2x}\left[\frac{e^{2y}}{2}\right]_{y=0}^{x} - e^x \left[\, e^y\, \right]_{y=0}^{x} \right\} dx$$

$$I = \int_{x=0}^{a} \left\{ \frac{e^{2x}}{2} (e^{2x} - 1) - e^{x} (e^{x} - 1) \right\} dx$$

$$= \int_{x=0}^{a} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x} \right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x} \right]_0^a$$

$$= \left(\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} \right) - \left(\frac{1}{8} - \frac{3}{4} + 1 \right)$$

$$I = \frac{e^{4a}}{8} - \frac{3\,e^{2a}}{4} + e^{a} - \frac{3}{8}$$

Thus $\boldsymbol{I = \frac{1}{8} (e^{4a} - 6e^{2a} + 8e^{a} - 3)}$

---

9. *Evaluate* $\int_0^{\pi/2} \int_0^{a \sin \theta} \int_0^{(a^2 - r^2)/a} r \, dr \, d\theta \, dz$

\>> $$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a \sin \theta} \int_{z=0}^{(a^2 - r^2)/a} r \, dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a \sin \theta} r \Big[z \Big]_{z=0}^{(a^2 - r^2)/a} dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a \sin \theta} r \cdot \frac{a^2 - r^2}{a} \, dr \, d\theta$$

$$= \frac{1}{a} \int_{\theta=0}^{\pi/2} \left\{ a^2 \left[\frac{r^2}{2} \right]_0^{a \sin \theta} - \left[\frac{r^4}{4} \right]_0^{a \sin \theta} \right\} d\theta$$

$$= \frac{1}{a} \int_{\theta=0}^{\pi/2} \left(\frac{a^4}{2} \sin^2 \theta - \frac{a^4}{4} \sin^4 \theta \right) d\theta$$

Using $\int_0^{\pi/2} \sin^n \theta \, d\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$ (*where n is even*)

$$I = \frac{1}{a}\left\{ \frac{a^4}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{a^4}{4} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right\} = \frac{\pi a^4}{8a}\left(1 - \frac{3}{8}\right) = \frac{5 \pi a^3}{64}$$

Thus **$I = 5 \pi a^3 / 64$**

---

10. *Evaluate* $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z - x^2}} dy\, dx\, dz$

>> $$I = \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \int_{y=0}^{\sqrt{4z - x^2}} dy\, dx\, dz$$

$$= \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \Big[y \Big]_0^{\sqrt{4z - x^2}} dx\, dz = \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \sqrt{4z - x^2}\, dx\, dz$$

Let $4z = a^2$ (*for convenience*) so that $2\sqrt{z} = a$

$$I = \int_{z=0}^{4} \int_{x=0}^{a} \sqrt{a^2 - x^2}\, dx\, dz$$

$$= \int_{z=0}^{4} \left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right]_{x=0}^{a} dz$$

$$= \int_{z=0}^{4} 0 + \frac{a^2}{2} (\sin^{-1} 1 - \sin^{-1} 0)\, dz$$

$$= \frac{\pi}{2} \int_{z=0}^{4} 2z\, dz$$

$$= \frac{\pi}{2} \Big[z^2 \Big]_0^4 = \frac{\pi}{2} (16 - 0) = 8\pi$$

Thus **$I = 8\pi$**

---

11. *Evaluate* $\int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^{x-y+z}\, dz\, dy\, dx$

>> $$I = \int_{x=0}^{\log 2} \int_{y=0}^{x} \int_{z=0}^{x+\log y} e^{x+y} \cdot e^{z}\, dz\, dy\, dx$$

$$= \int_{x=0}^{\log 2} \int_{y=0}^{x} e^{x+y} \Big[e^{z} \Big]_{z=0}^{x+\log y} dy\, dx$$

$$= \int_{x=0}^{\log 2} \int_{y=0}^{x} e^{x+y} [\, e^{x+\log y} - 1 \,]\, dy\, dx. \text{ But } e^{\log y} = y$$

$$= \int_{x=0}^{\log 2} \int_{y=0}^{x} [\, e^{2x} \cdot y e^{y} - e^{x} \cdot e^{y} \,]\, dy\, dx$$

$$= \int_{x=0}^{\log 2} \Big[e^{2x} (\, y e^{y} - e^{y} \,) - e^{x} \cdot e^{y} \Big]_{y=0}^{x} dx$$

$$= \int_{x=0}^{\log 2} \Big[e^{2x} \big\{ (\, x e^{x} - e^{x} \,) - (\, 0 - 1 \,) \big\} - e^{x} (\, e^{x} - 1 \,) \Big] dx$$

$$= \int_{x=0}^{\log 2} (\, x e^{3x} - e^{3x} + e^{2x} - e^{2x} + e^{x} \,)\, dx$$

$$= \int_{0}^{\log 2} (\, x e^{3x} - e^{3x} + e^{x} \,)\, dx$$

$$= \left[x \cdot \frac{e^{3x}}{3} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^{x} \right]_0^{\log 2}$$

$$= \left[\frac{x e^{3x}}{3} - \frac{4 e^{3x}}{9} + e^{x} \right]_0^{\log 2}$$

$$= \left(\frac{\log 2 \cdot e^{3 \log 2}}{3} - 0 \right) - \frac{4}{9} (\, e^{3 \log 2} - 1 \,) + (\, e^{\log 2} - 1 \,)$$

$$= \frac{8\log 2}{3} - \frac{4}{9}(8-1) + (2-1)$$

$$= \frac{8\log 2}{3} - \frac{28}{9} + 1 = \frac{8\log 2}{3} - \frac{19}{9}$$

Thus $\quad \mathbf{I = \frac{\log 256}{3} - \frac{19}{9}}$

---

12. *Evaluate* $\int\limits_{0}^{1} \int\limits_{0}^{1-x} \int\limits_{0}^{1-x-y} \frac{dz\,dy\,dx}{(1+x+y+z)^3}$

Let $\quad I = \int\limits_{x=0}^{1} \int\limits_{y=0}^{1-x} \int\limits_{z=0}^{1-x-y} \frac{dz\,dy\,dx}{(1+x+y+z)^3}$

$$= \int\limits_{x=0}^{1} \int\limits_{y=0}^{1-x} \left[\frac{-1}{2(1+x+y+z)^2} \right]_{z=0}^{1-x-y} dy\,dx$$

$$= \int\limits_{x=0}^{1} \int\limits_{y=0}^{1-x} \left[-\frac{1}{8} + \frac{1}{2(1+x+y)^2} \right] dy\,dx$$

$$= \int\limits_{x=0}^{1} \left[-\frac{1}{8}y - \frac{1}{2(x+y+1)} \right]_{y=0}^{1-x} dx$$

$$= \int\limits_{x=0}^{1} \left[-\frac{1}{8}(1-x) - \frac{1}{4} + \frac{1}{2(x+1)} \right] dx$$

$$= \int\limits_{x=0}^{1} \left[-\frac{3}{8} + \frac{x}{8} + \frac{1}{2(x+1)} \right] dx$$

$$= \left[-\frac{3x}{8} + \frac{x^2}{16} + \frac{1}{2}\log(x+1) \right]_{x=0}^{1}$$

$$= -\frac{3}{8} + \frac{1}{16} + \frac{1}{2}\log 2 = \frac{-5}{16} + \log\sqrt{2}$$

Thus $\quad \mathbf{I = \log\sqrt{2} - (5/16)}$

5.22 Evaluation of $\iint\limits_R f(x, y)\,dx\,dy$ over the specific region R

We need to draw the befitting figure from the given description to identify the specific region R. We have to then express

$$I = \iint\limits_R f(x, y)\,dx\,dy = \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} f(x, y)\,dy\,dx \qquad \ldots (1)$$

or

$$I = \iint\limits_R f(x, y)\,dx\,dy = \int_{y=c}^{d} \int_{x=x_1(y)}^{x_2(y)} f(x, y)\,dx\,dy \qquad \ldots (2)$$

I is obtained by the evaluation of (1) or (2)

Remark : Carefully take a note of the content in article 5.21

5.23 Evaluation of double integral by changing the order of integration

- Given the integral in either of the forms as in article 2.22, say (1) we have to identify the region of integration R by writing the figure (*ie., the converse situation*) and express (1) in the form (2).
- The evaluation of (2) will be the value of (1) on changing the order of integration. This can be vice versa also.
- The advantage of this procedure is that, some times the double integral which is difficult to be evaluated in the existing form becomes easy for evaluation on changing the order of integration.

5.24 Evaluation of double integral by changing into polar form

- Given a double integral with limits we use the well known polar form of substitution $x = r\cos\theta$, $y = r\sin\theta$. This will give us $x^2 + y^2 = r^2$, $y/x = \tan\theta$ and it should be noted that $dx\,dy = J\,dr\,d\theta$ where J is the Jacobian of the transformation given by

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

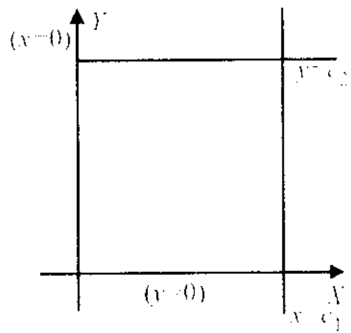
- Thus $dx\,dy = r\,dr\,d\theta$ and we need to change the limits of integration to r, θ suitably for the purpose of evaluation.
- The method might be advantageous if the terms of the form $x^2 + y^2$ are involved in $f(x, y)$ and terms like $\sqrt{a^2 - y^2}$, $\sqrt{a^2 - x^2}$ etc. are involved in limits.

Note : *Some of the important and standard curves along with their equations and shape is given below as it will be highly useful for working problems.*

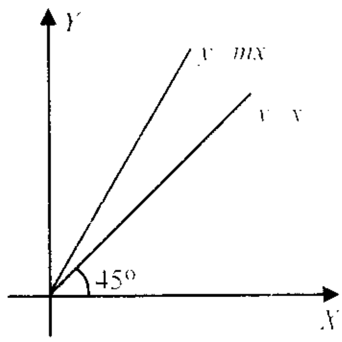
1. ***Straight lines***

(i) $x = 0$ and $y = 0$ are respectively the equations of y and x axis.

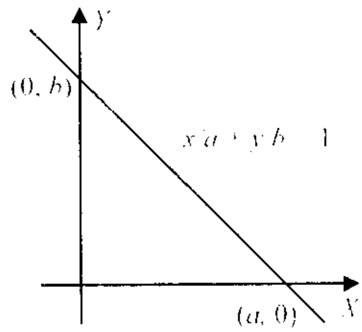
(ii) $x = c_1$ and $y = c_2$ are respectively the equations of a line parallel to y-axis and a line parallel to x axis.



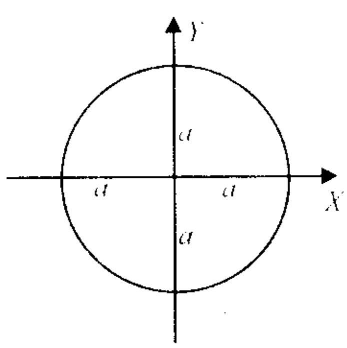
(iii) $y = mx$ is a straight line passing through the origin and in particular $y = x$ is a straight line passing through the origin subtending an angle 45^0 with the x axis.



(iv) $x/a + y/b = 1$ is a straight line having x intercept a and y *intercept* b *ie.*, a straight line passing through $(a, 0)$ and $(0, b)$

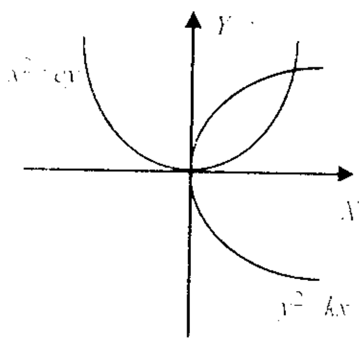


2. ***Circle*** $x^2 + y^2 = a^2$ is a circle with centre origin and radius a.



3. ***Parabola*** $y^2 = kx$ is symmetrical about the *x - axis.*

$x^2 = cy$ is symmetrical about the y - axis.



5.25 Area, Volume and Surface area

1. $\iint_R dx\,dy =$ Area of the region R in the cartesian form.
2. $\iint_R r\,dr\,d\theta =$ Area of the region R in the polar form.
3. $\iiint_V dx\,dy\,dz =$ Volume of a solid.
4. If $z = f(x, y)$ be the equation of a surface S then the surface area is given by

$$\iint_A \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dx\,dy$$

where A is the region representing the projection of S on the xoy plane.

5. **Volume** of a solid (*in polars*) obtained by the revolution of a curve enclosing an area A about the initial line is given by

$$V = \iint_A 2\pi r^2 \sin\theta\, dr\, d\theta$$

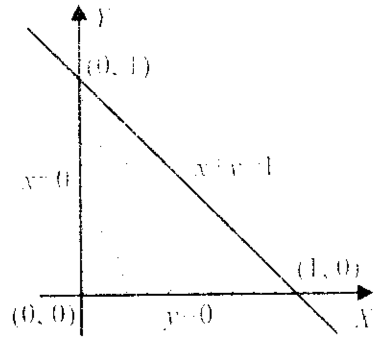
WORKED PROBLEMS

Type-1 : *Evaluation over a given region*

13. *Evaluate* $\iint_R xy\,dx\,dy$ *where R is the region bounded by the coordinate axes and the line* $x + y = 1$

>> R is the region bounded by $x = 0$, $y = 0$ being the coordinate axes and $x + y = 1$ being a straight line through $(1, 0)$ and $(0, 1)$.

Shaded portion in the figure is the region R.
From the figure we have



$$I = \iint_R xy\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} xy\,dy\,dx$$

or $$\int_{y=0}^{1}\int_{x=0}^{1-y} xy\,dx\,dy$$

$$I = \int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_{y=0}^{1-x} dx = \int_{x=0}^{1} \frac{x}{2}(1-x)^2\,dx$$

$$I = \frac{1}{2}\int_{x=0}^{1}(x-2x^2+x^3)\,dx = \frac{1}{2}\left[\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\right]_0^1$$

$$I = \frac{1}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = \frac{1}{24}$$

Thus **$I = 1/24$**

Remark : *Given the integral* $\int_{x=0}^{1}\int_{y=0}^{1-x} xy\,dy\,dx$ *we can write the figure to identify the region of integration. This being the region bounded by* $y = 0$ *(x-axis),* $y = 1-x$ *or* $x+y = 1$ *a line passing through the points* $(1, 0)$ *and* $(0, 1)$ *embedded between the lines* $x = 0$, $x = 1$. $x = 0$ *to 1 being the horizontal strip can be changed to vertical strip* $y = 0$ *to 1 (constant limits)* $y = 0$ *to* $(1-x)$ *being the vertical strip can be changed to horizontal strip* $x = 0$ *to* $1-y$ *(variable limits). This is the principle of changing the order of integration of a given double integral.*

---

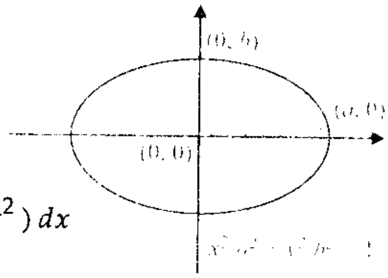
14. *Evaluate* $\iint y\,dx\,dy$ *over the region bounded by the first quadrant of the ellipse* $x^2/a^2+y^2/b^2 = 1$

>> Shaded portion in the figure is the region (R) of integration. We observe that x varies from 0 to a and we need to express $x^2/a^2+y^2/b^2 = 1$ in the form $y = f(x)$.

ie., $$y^2/b^2 = 1-(x^2/a^2) = (a^2-x^2)/a^2$$

or $$y = (b/a)\sqrt{a^2-x^2}$$

Since $y = 0$ is the equation of x axis, we can say that y varies from 0 to $(b/a)\sqrt{a^2-x^2}$



$$\therefore \quad \iint_R y\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{(b/a)\sqrt{a^2-x^2}} y\,dy\,dx \qquad \ldots (1)$$

$$= \int_{x=0}^{a} \left[\frac{y^2}{2}\right]_{y=0}^{(b/a)\sqrt{a^2-x^2}} dx = \frac{b^2}{2a^2} \int_{x=0}^{a} (a^2-x^2)\,dx$$

$$= \frac{b^2}{2a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a = \frac{b^2}{2a^2}\left[\left(a^3 - \frac{a^3}{3}\right) - 0\right] = \frac{a b^2}{3}$$

Thus $\quad I = ab^2/3$

Note : *From the figure, on a similar argument we can also have*

$$\iint_R y\,dx\,dy = \int_{y=0}^{b} \int_{x=0}^{(a/b)\sqrt{b^2-y^2}} y\,dx\,dy = \frac{ab^2}{3} \qquad \ldots (2)$$

Remark : *Given the double integral in the form (1) writing the same in the form as in (2) with the help of the figure is the change of order of integration.*

---

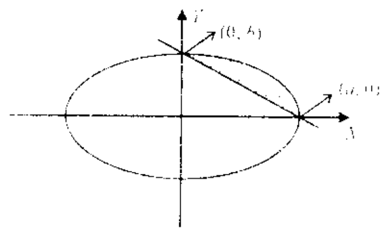
15. *Evaluate* $\iint xy\,dx\,dy$ *taken over the region bounded by* $x^2/a^2 + y^2/b^2 = 1$ *and* $x/a + y/b = 1$

>> $\quad x$ varies from 0 to a

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ or } y = \frac{b}{a}(a-x)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

or $\quad y = \dfrac{b}{a}\sqrt{a^2-x^2}$



$$I = \iint_R xy\,dx\,dy = \int_{x=0}^{a} \int_{b(a-x)/a}^{(b/a)\sqrt{a^2-x^2/a}} xy\,dy\,dx$$

$$I = \int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{b(a-x)/a}^{(b/a)\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_{x=0}^{a} x\left\{\frac{b^2}{a^2}(a^2-x^2) - \frac{b^2}{a^2}(a-x)^2\right\} dx$$

$$I = \frac{b^2}{2a^2} \int_0^a (a^2 x - x^3 - a^2 x + 2a x^2 - x^3)\, dx$$

$$= \frac{b^2}{2a^2} \int_0^a 2(ax^2 - x^3)\, dx$$

$$= \frac{b^2}{a^2}\left[a\frac{x^3}{3} - \frac{x^4}{4}\right]_0^a = \frac{b^2}{a^2}\left(\frac{a^4}{3} - \frac{a^4}{4}\right) = \frac{a^2 b^2}{12}$$

Thus **$I = a^2 b^2 / 12$**

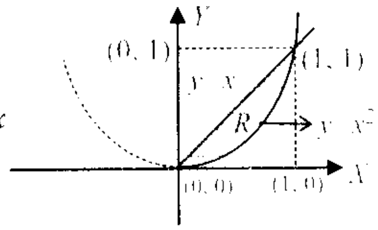
---

16. *Evaluate* $\iint xy(x+y)\, dy\, dx$ *taken over the area between* $y = x^2$ *and* $y = x$

>> Now $x^2 = x$ or $x(x-1) = 0 \Rightarrow x = 0,\ x = 1$. This gives $y = 0,\ y = 1$ and hence the two curves intersect at the points $(0, 0)$ and $(1, 1)$

$$I = \iint_R xy(x+y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x^2 y + xy^2)\, dy\, dx$$



$$= \int_{x=0}^{1} \left\{ x^2\left[\frac{y^2}{2}\right]_{y=x^2}^{x} + x\left[\frac{y^3}{3}\right]_{y=x^2}^{x} \right\} dx$$

$$= \int_{x=0}^{1} \left(\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3}\right) dx$$

$$= \left[\frac{x^5}{10} - \frac{x^7}{14} + \frac{x^5}{15} - \frac{x^8}{24}\right]_{x=0}^{1}$$

$$= \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24} = \frac{3}{56}$$

Thus **$I = 3/56$**

Note : *We can also write I in the form*

$$I = \int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} xy\,(x+y)\,dx\,dy \;;\; I = \frac{3}{56}$$

---

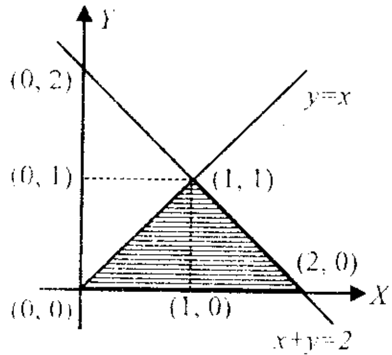
17. *Evaluate* $\iint_R x^2 y\,dx\,dy$ *where R is the region bounded by the lines* $y = x$, $y+x = 2$ *and* $y = 0$

>> The lines $y = x$ and $y + x = 2$ intersect at $(1, 1)$

$$I = \iint_R x^2 y\,dx\,dy = \int_{y=0}^{1} \int_{x=y}^{2-y} x^2 y\,dx\,dy = \int_{y=0}^{1} y\left[\frac{x^3}{3}\right]_{x=y}^{2-y} dy$$

$$I = \frac{1}{3}\int_{y=0}^{1} y\left\{(2-y)^3 - y^3\right\} dy = \frac{1}{3}\int_{y=0}^{1} y\,(8 - 12y + 6y^2 - y^3 - y^3)\,dy$$

$$I = \frac{1}{3}\int_{y=0}^{1} (8y - 12y^2 + 6y^3 - 2y^4)\,dy$$



$$I = \frac{1}{3}\left[4y^2 - 4y^3 + \frac{3}{2}y^4 - 2\frac{y^5}{5}\right]_{y=0}^{1}$$

$$= \frac{1}{3}\left(4 - 4 + \frac{3}{2} - \frac{2}{5}\right) = \frac{11}{30}$$

Thus $\mathbf{I = 11/30}$

Note : *Alternative form of I*

$$I = \int_{x=0}^{1} \int_{y=0}^{x} x^2 y\,dy\,dx + \int_{x=1}^{2} \int_{y=0}^{2-x} x^2 y\,dy\,dx \;;\; I = \frac{11}{30}$$

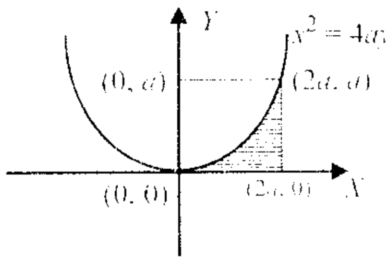
18. *Evaluate* $\iint_R xy\,dx\,dy$ *where R is the region bounded by the x-axis, ordinate* $x = 2a$ *and the curve* $x^2 = 4ay$.

>> $x^2 = 4ay$ is a parabola symmetrical about the y-axis. The point of intersection of this curve with $x = 2a$ is to be found.

Hence $(2a)^2 = 4ay$ or $4a^2 = 4ay \;\therefore\; y = a$

The point of intersection is $(2a, a)$

$$I = \iint_R xy\,dx\,dy = \int_{x=0}^{2a} \int_{y=0}^{x^2/4a} xy\,dy\,dx$$



$$I = \int_{x=0}^{2a} x\left[\frac{y^2}{2}\right]_{y=0}^{x^2/4a} dx = \frac{1}{2}\int_{x=0}^{2a} x\left[\frac{x^4}{16a^2}\right] dx$$

$$I = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{64a^6}{32 \cdot 6a^2} = \frac{a^4}{3}$$

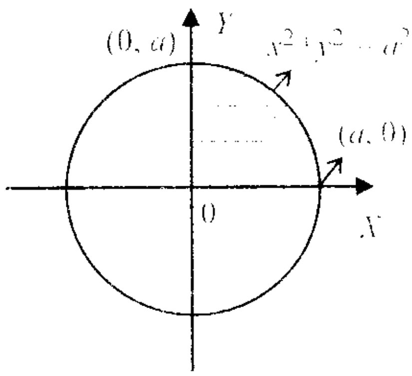
Thus $\mathbf{I = a^4 / 3}$

Note : *Alternative form of* $I = \int_{y=0}^{a} \int_{x=\sqrt{4ay}}^{2a} xy\,dx\,dy$; $I = \frac{a^4}{3}$

---

19. *Evaluate* $\iint xy\,dx\,dy$ *over the positive quadrant of the circle* $x^2 + y^2 = a^2$.

$$I = \iint_R xy\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$



$$I = \int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{a} x(a^2 - x^2)\,dx = \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{a}$$

$$I = \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8}$$

Thus $\mathbf{I = a^4 / 8}$

Note : *Alternative form of* I : $\int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} xy\,dx\,dy$; $I = \frac{a^4}{8}$

Type-2. *Evaluation of a double integral by changing the order of integration*

- The procedure is illustrated in the article 5.23 and in every problem (13 to 19) the alternative form of the double integral is exactly the integral by changing the order of integration.
- We complete the problem by evaluating the new form of the double integral.

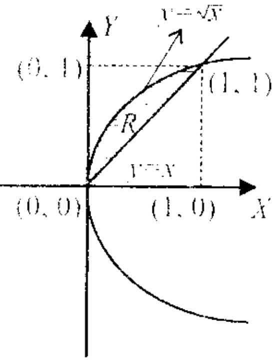
WORKED PROBLEMS

20. *Evaluate* $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$ *by changing the order of integration.*

>> $$I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy\, dy\, dx$$

We need to first identify the region of integration R bounded by the curves $y = x$, $y = \sqrt{x}$ between the lines $x = 0$, $x = 1$. We shall find the points of intersection of $y = x$ and $y = \sqrt{x}$ by equating their R.H.S

ie., $x = \sqrt{x} \Rightarrow x^2 = x$ or $x(x-1) = 0$ *ie.,* $x = 0, 1$. This will give us $y = 0$, $y = 1$ and hence the points of intersection are $(0, 0)$ and $(1, 1)$. Further we know that $y = x$ is a straight line passing through the origin making an angle 45° with the x-axis and $y = \sqrt{x}$ or $y^2 = x$ is a parabola symmetrical about the x-*axis*. The befitting figure indicating R is given.



On changing the order of integration we must have constant limits for y and variable limits for x. From the figure we observe that y varies from 0 to 1 and x varies from y^2 ($\because y = \sqrt{x}$) to y

[*It should be noted that* $y = x$ *and* $\sqrt{x}$ *are the lower and upper parts of the boundary of R whereas on changing the order* $x = y^2$ *and* $x = y$ *represent the left and right parts of the boundary of R*]

Thus we have on changing the order of integration

$$I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy\, dx\, dy$$

$$= \int_{y=0}^{1} y \left[\frac{x^2}{2} \right]_{x=y^2}^{y} dy = \frac{1}{2} \int_{y=0}^{1} y\,(y^2 - y^4)\, dy$$

$$I = \frac{1}{2}\int_0^1 (y^3 - y^5)\, dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1 = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{24}$$

Thus $\boldsymbol{I = 1/24}$

Remark : *Referring to Problem-1 it may be seen that we have obtained the same answer by direct evaluation.*

---

21. *Change the order of the integration and hence evaluate* $\int_0^1 \int_{\sqrt{y}}^1 dx\ dy$

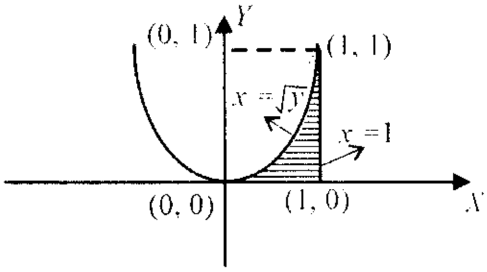
>> Let $I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} dx\ dy$

On changing the order of integration,

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} dy\ dx \quad (x = \sqrt{y} \Rightarrow x^2 = y)$$

$$= \int_{x=0}^{1} [\,y\,]_0^{x^2}\, dx$$

$$= \int_{x=0}^{1} x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

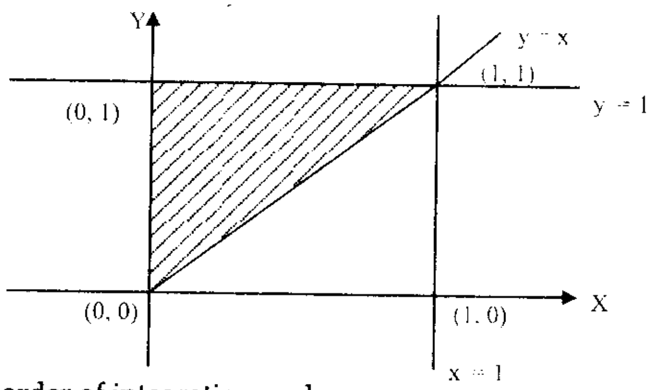


Thus $\boldsymbol{I = 1/3}$

---

22. *Evaluate by change of order of integration* $\int_0^1 \int_x^1 \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$

>> Let $I = \int_{x=0}^{1} \int_{y=x}^{1} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$



On changing the order of integration we have,

$$I = \int_{y=0}^{1} \int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

Put $x^2+y^2 = t \;\therefore\; 2x\,dx = dt$ or $x\,dx = dt/2$

We have $\int \frac{x}{\sqrt{x^2+y^2}}\, dx$ reducing to $\frac{1}{2}\int \frac{dt}{\sqrt{t}} = \sqrt{t}$

$$\therefore \quad I = \int_{y=0}^{1} \left[\sqrt{x^2+y^2}\right]_{x=0}^{y} dy$$

$$= \int_{y=0}^{1} (\sqrt{2}\,y - y)\, dy = \int_{0}^{1} (\sqrt{2}-1)\, y\, dy$$

$$= (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1 = \frac{\sqrt{2}-1}{2}$$

Thus $\boldsymbol{I = (\sqrt{2}-1)/2}$

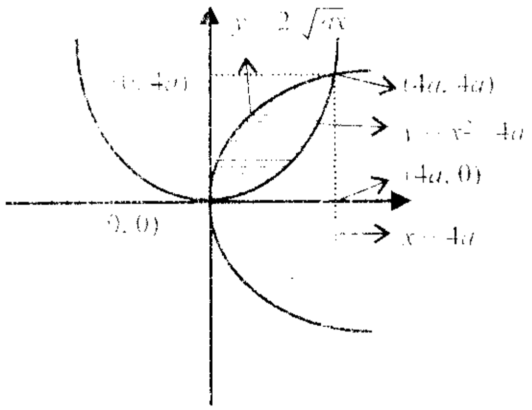
---

23. *Change the order of integration and hence evaluate* $\int_{0}^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx$

>> We have, $I = \int_{x=0}^{4a} \int_{y=x^2/4a}^{y=2\sqrt{ax}} xy\, dy\, dx$

We have $\frac{x^2}{4a} = 2\sqrt{ax}$ or $x^4 = 64a^3 x$

i.e., $x(x^3 - 64a^3) = 0 \quad \Rightarrow \quad x = 0$ and $x = 4a$



From $y = x^2/4a$ we get $y = 0$ and $y = 4a$

Thus the points of intersection of the parabolas $y = x^2/4a$ and $y = 2\sqrt{ax}$ are $(0, 0)$ and $(4a, 4a)$

On changing the order of integration we have y varying from 0 to $4a$ and

x varying from $y^2/4a$ ($\because y = 2\sqrt{ax}$) to $2\sqrt{ay}$ ($\because y = x^2/4a$)

Now $$I = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} xy \, dx \, dy$$

$$= \int_{y=0}^{4a} y \left[\frac{x^2}{2} \right]_{x = y^2/4a}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2} \int_{y=0}^{4a} y \left[4ay - \frac{y^4}{16a^2} \right] dy$$

$$= \frac{1}{2} \left[4a \frac{y^3}{3} - \frac{1}{16a^2} \frac{y^6}{6} \right]_{y=0}^{4a}$$

$$= \frac{1}{2} \left[4a \left(\frac{64a^3}{3} \right) - \frac{1}{96a^2} (4096a^6) \right]$$

$$= \frac{1}{2} \left[\frac{256a^4}{3} - \frac{128a^4}{3} \right] = \frac{64a^4}{3}$$

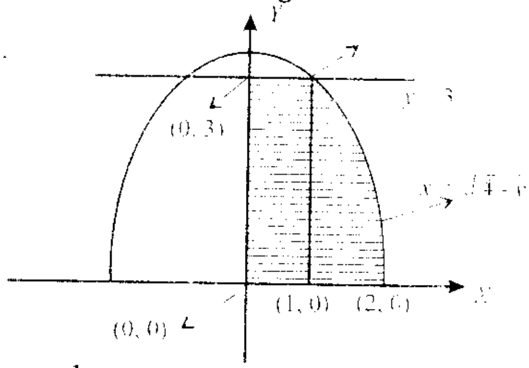
Thus **$I = 64a^4/3$**

---

24. *Change the order of integration and evaluate* $\int_0^3 \int_0^{\sqrt{4-y}} (x+y)\, dx\, dy$

>> Let $I = \int_{y=0}^{3} \int_{x=0}^{\sqrt{4-y}} (x+y)\, dx\, dy$

The points of intersection of the parabola $x = \sqrt{4-y}$ or $x^2 = 4-y$ with $y = 0$ are $(\pm 2, 0)$ and with $y = 3$ are $(\pm 1, 0)$. Since y varies from 0 to 3 the points for consideration are (2, 0) and (1, 0). The region is shown in the figure.



On changing the order we have,

$$I = \int_{x=0}^{1} \int_{y=0}^{3} (x+y)\, dy\, dx + \int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\, dy\, dx = I_1 + I_2 \text{ (say)}$$

Now $I_1 = \int_{x=0}^{1} \left[xy + \frac{y^2}{2} \right]_{y=0}^{3} dx$

$$= \int_{x=0}^{1} \left(3x + \frac{9}{2} \right) dx = \left[\frac{3x^2}{2} + \frac{9x}{2} \right]_0^1 = \frac{3}{2} + \frac{9}{2} = 6$$

Next $I_2 = \int_{x=1}^{2} \left[xy + \frac{y^2}{2} \right]_{y=0}^{4-x^2} dx$

$$= \frac{1}{2} \int_{x=1}^{2} [\, 2x(4-x^2) + (4-x^2)^2 \,]\, dx$$

$$= \frac{1}{2} \int_{x=1}^{2} [\, x^4 - 2x^3 - 8x^2 + 8x + 16 \,]\, dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5}-\frac{x^4}{2}-\frac{8x^3}{3}+4x^2+16x\right]_1^2$$

$$= \frac{1}{2}\left[\frac{31}{5}-\frac{15}{2}-\frac{56}{3}+12+16\right] = \frac{241}{60}$$

Now $I = I_1 + I_2 = 6 + \frac{241}{60} = \frac{601}{60}$

Thus $\boldsymbol{I = 601/60}$

---

25. *Evaluate* $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ *by changing the order of integration.*

>> $$I = \int_{x=0}^{1} \int_{y=x^2}^{2-x} xy\, dy\, dx$$

To identify the region of integration R, let us find the points of intersection of the curves $y = x^2$ and $y = 2-x$

ie., to solve $x^2 = 2-x$

or $x^2+x-2 = 0$ *ie.,* $(x-1)(x+2) = 0 \Rightarrow x = 1,\ x = -2$

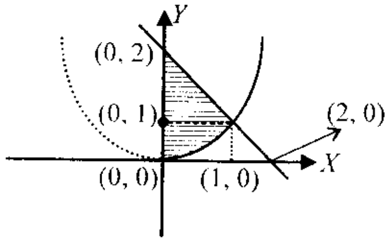
$\therefore$ from $y = x^2$ *(or* $y = 2-x$*)* we obtain $y = 1,\ y = 4$.

Thus $(1, 1)$ and $(-2, 4)$ are the points of intersection. Since the region is bounded by $x = 0$ and $x = 1$ the point $(-2, 4)$ will be out of consideration. Further $y = x^2$ is a parabola symmetrical about the y-axis and $y = 2-x$ or $x+y = 2$ or $x/2+y/2 = 1$ is a straight line passing through $(2, 0)$ and $(0, 2)$. With these findings the region is indicated in the figure.

On changing the order we must have constant limits for y and variable limits for x to cover the same region.

From the figure we can make out that the same will be in two parts.

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx \, dy = I_1 + I_2 \ (say)$$



Now $I_1 = \int_{y=0}^{1} y \left[\frac{x^2}{2} \right]_{x=0}^{\sqrt{y}} dy = \int_{y=0}^{1} \frac{y^2}{2} dy = \left[\frac{y^3}{6} \right]_0^1 = \frac{1}{6}$

$$I_2 = \int_{y=1}^{2} y \left[\frac{x^2}{2} \right]_{x=0}^{2-y} dy = \frac{1}{2} \int_{y=1}^{2} y(2-y)^2 dy$$

ie., $= \frac{1}{2} \int_{y=1}^{2} (4y - 4y^2 + y^3) \, dy = \frac{1}{2} \left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4} \right]_{y=1}^{2}$

ie., $= \frac{1}{2} \left[2(4-1) - \frac{4}{3}(8-1) + \frac{1}{4}(16-1) \right] = \frac{5}{24}$

$\therefore \quad I = I_1 + I_2 = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$

Thus $\quad I = 3/8$

---

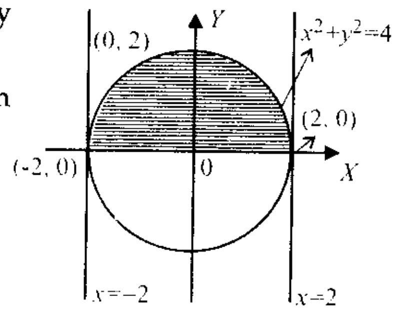
26. *Evaluate* $\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} (2-x) \, dy \, dx$ *by changing the order of integration.*

>> We have $I = \int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} (2-x) \, dy \, dx$

Here, $y = \sqrt{4-x^2}$ or $x^2 + y^2 = 4$

This is a circle with centre origin and radius 2. $y = 0$ to $\sqrt{4-x^2}$ is the upper half of the circle being bounded by the lines $x = -2$ and 2.

On changing the order we must have y varying from 0 to 2 and x from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$



$\therefore \quad I = \int_{y=0}^{2} \int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2-x) \, dx \, dy$

$$I = \int_{y=0}^{2} \left[2x - \frac{x^2}{2} \right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$I = \int_{y=0}^{2} \left[2 \cdot 2\sqrt{4-y^2} - 0 \right] dy = 4 \int_{y=0}^{2} \sqrt{2^2 - y^2}\, dy$$

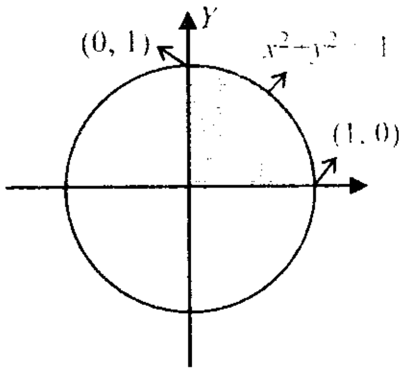
$$I = 4 \left[\frac{y\sqrt{4-y^2}}{2} + \frac{2^2}{2} \sin^{-1} \frac{y}{2} \right]_0^2 = 4\,(0 + 2\sin^{-1} 1) = 8 \cdot \frac{\pi}{2} = 4\pi$$

Thus $\boldsymbol{I = 4\pi}$

---

27. *Change the order of integration and hence evaluate* $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dx\, dy$

>> $I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y^2\, dy\, dx$



On changing the order of integration we have from the figure

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2\, dx\, dy$$

$$I = \int_{y=0}^{1} y^2 \left[x \right]_{x=0}^{\sqrt{1-y^2}} dy = \int_{y=0}^{1} y^2 \sqrt{1-y^2}\, dy$$

Put $y = \sin\theta \;\therefore\; dy = \cos\theta\, d\theta$ and θ varies from 0 to $\pi/2$

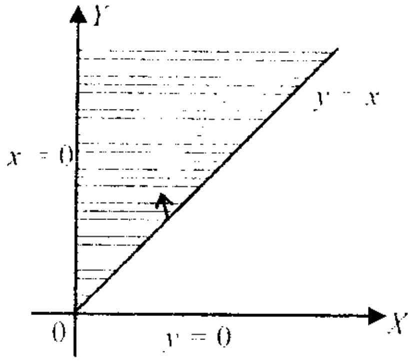
$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta = \frac{(1)(1)}{(4)(2)} \cdot \frac{\pi}{2} = \frac{\pi}{16}$$ by reduction formula.

Thus $\boldsymbol{I = \pi/16}$

---

28. *Change the order of integration and evaluate* $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$.

>> $$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y} \, dy \, dx$$



On changing the order we must have
$y = 0$ to ∞ and $x = 0$ to y

$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y} \, dx \, dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y} [\, x \,]_0^y \, dy$$

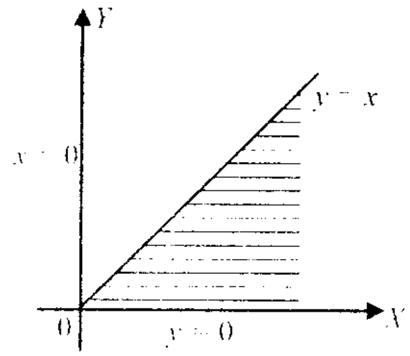
$$I = \int_{y=0}^{\infty} \frac{e^{-y}}{y} \cdot y \, dy = \int_{y=0}^{\infty} e^{-y} \, dy = -[\, e^{-y} \,]_0^{\infty} = 1$$

Thus $I = 1$

---

29. *Evaluate* $\int_0^\infty \int_0^x x \, e^{-x^2/y} \, dy \, dx$ *by changing the order of integration.*

>> $$I = \int_{x=0}^{\infty} \int_{y=0}^{x} x \, e^{-x^2/y} \, dy \, dx$$



The region is as shown in the figure.
On changing the order of integration
we must have $y = 0$ to ∞ ; $x = y$ to ∞

$$I = \int_{y=0}^{\infty} \int_{x=y}^{\infty} x e^{-x^2/y} \, dx \, dy$$

Put $\frac{x^2}{y} = t \quad \therefore \quad \frac{2x}{y} dx = dt$ or $x \, dx = y \, dt/2$

Also when $x = y$, $t = y$ and when $x = \infty$, $t = \infty$

$$\therefore \quad I = \int_{y=0}^{\infty} \int_{t=y}^{\infty} e^{-t} \frac{y}{2} \, dt \, dy$$

$$= \int_{y=0}^{\infty} \frac{y}{2} \left[-e^{-t} \right]_{t=y}^{\infty} dy$$

$$= \frac{1}{2} \int_{y=0}^{\infty} y\, e^{-y}\, dy.$$

Applying Bernoulli's rule,

$$I = \frac{1}{2}\left\{ [\, y\,(-e^{-y})\,]_{y=0}^{\infty} - [\,(1)\,(e^{-y})\,]_{y=0}^{\infty} \right\}$$

$$= \frac{1}{2}[\,0-(0-1)\,] = \frac{1}{2}$$

Thus $\boldsymbol{I = 1/2}$

---

30. *Show that* $\displaystyle\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy = \frac{\pi a}{4}$

>> **Note** : *We work the problem directly and also by changing the order of integration. The advantage by changing the order can be clearly felt.*

Method-1 *(Direct evaluation)*

$$I = \int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2}\, dx\, dy$$

$$= \int_{y=0}^{a} \frac{1}{2}\left[\, \log(x^2+y^2)\,\right]_{x=y}^{a} dy$$

$$= \frac{1}{2} \int_{y=0}^{a} [\, \log(a^2+y^2) - \log(2y^2)\,]\, dy$$

$$= \frac{1}{2} \int_{y=0}^{a} [\, \log(a^2+y^2) - \log 2 - 2\log y\,]\, dy$$

$$= \frac{1}{2} \int_{y=0}^{a} \log(a^2+y^2)\cdot 1\, dy - \frac{1}{2}\log 2\,[\, y\,]_0^a - \int_{y=0}^{a} \log y \cdot 1\, dy$$

$$= \frac{1}{2}\left\{ [\, \log(a^2+y^2)\cdot y\,]_{y=0}^{a} - \int_0^a y\cdot\frac{2y}{a^2+y^2}\, dy \right\} - \frac{1}{2}a\log 2 - [\, y\log y - y\,]_0^a$$

$$= \frac{1}{2}\left\{ a\log(2a^2) - 2\int_0^a \left(1 - \frac{a^2}{a^2+y^2}\right) dy \right\} - \frac{a}{2}\log 2 - (\, a\log a - a\,)$$

$$= \frac{a}{2}(\log 2 + 2\log a) - \left[y - a^2 \cdot \frac{1}{a}\tan^{-1}(y/a)\right]_0^a - \frac{a}{2}\log 2 - a\log a + a$$

$$= -a + a(\tan^{-1} 1 - \tan^{-1} 0) + a = a \cdot \pi/4 = \pi a/4$$

Thus $I = \pi a/4$

Method-2 *(By changing the order of integration)*

The region bounded by the curves $x = y$, $x = a$ embedded between the lines $y = 0$, $y = a$ is shown in the figure

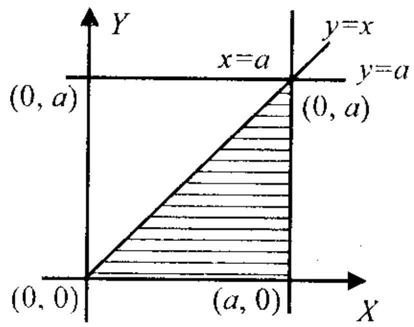
On changing the order x varies from 0 to a and y varies from 0 to x.

$$I = \int_{x=0}^{a} \int_{y=0}^{x} x \cdot \frac{1}{x^2 + y^2}\, dy\, dx$$

$$= \int_{x=0}^{a} x \cdot \frac{1}{x}\left[\tan^{-1}(y/x)\right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{a} (\tan^{-1} 1 - \tan^{-1} 0)\, dx$$

$$= \int_0^a \frac{\pi}{4}\, dx = \frac{\pi}{4}[x]_0^a = \frac{\pi a}{4}$$



Thus $I = \pi a / 4$

---

31. *Change the order of integration in* $\int_0^1 \int_{\sqrt{y}}^{2-y} xy\, dx\, dy$ *and evaluate.*

>> $I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{2-y} xy\, dx\, dy$

First let us find the points of intersection of the curves $x = \sqrt{y}$ and $x = 2 - y$

ie., $\sqrt{y} = 2 - y$ or $y = (2-y)^2$ or $y^2 - 5y + 4 = 0$

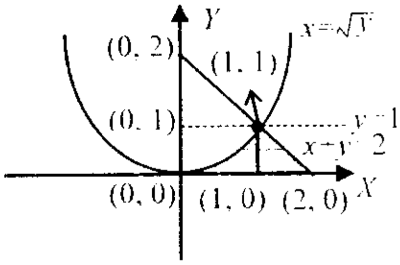
ie., $(y-1)(y-4) = 0 \Rightarrow y = 1$ and 4 $\therefore$ $x = \pm 1, \pm 2$

The points of intersection are $(1, 1)(-1, 1)(2, 4)(-2, 4)$

Since y varies from 0 to 1 the point $(1, 1)$ is of consideration. $x = \sqrt{y}$ or $x^2 = y$ is a parabola symmetrical about the y-axis and $x = 2 - y$ or $x + y = 2$ is a line passing through $(2, 0)$ and $(0, 2)$. The region is as shown in the figure.

The integral on changing the order consist of two parts.

$$I = \int_{x=0}^{1}\int_{y=0}^{x^2} xy\,dy\,dx + \int_{x=1}^{2}\int_{y=0}^{2-x} xy\,dy\,dx = I_1 + I_2\ (say)$$



$$I_1 = \int_{x=0}^{1}\int_{y=0}^{x^2} xy\,dy\,dx = \int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_{y=0}^{x^2} dx$$

ie.,

$$= \int_{x=0}^{1}\frac{x^5}{2}\,dx$$

$$I_1 = \left[\frac{x^6}{12}\right]_0^1 = \frac{1}{12}$$

$$I_2 = \int_{x=1}^{2}\int_{y=0}^{2-x} xy\,dy\,dx = \int_{x=1}^{2} x\left[\frac{y^2}{2}\right]_{y=0}^{2-x} dx$$

$$= \frac{1}{2}\int_{x=1}^{2} x\,(2-x)^2\,dx$$

$$= \frac{1}{2}\int_{x=1}^{2}(4x - 4x^2 + x^3)\,dx$$

$$= \left[x^2\right]_1^2 - 2\left[\frac{x^3}{3}\right]_1^2 + \left[\frac{x^4}{8}\right]_1^2$$

$$I_2 = (4-1) - \frac{2}{3}(8-1) + \frac{1}{8}(16-1) = 3 - \frac{14}{3} + \frac{15}{8} = \frac{5}{24}$$

Hence $I = I_1 + I_2 = \frac{1}{12} + \frac{5}{24} = \frac{7}{24}$

Thus **$I = 7/24$**

---

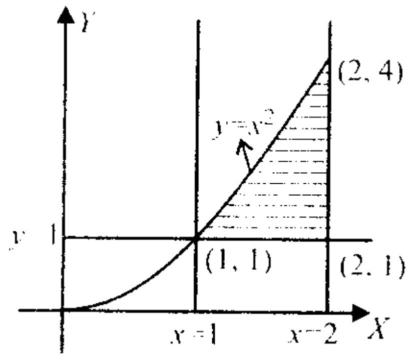
32. *Evaluate by changing the order of integration* $\int_{1}^{2}\int_{1}^{x^2}(x^2 + y^2)\,dy\,dx$

\>> $$I = \int_{x=1}^{2}\int_{y=1}^{x^2}(x^2 + y^2)\,dy\,dx$$

The points of intersection of $y = 1$ with $x = 1$, $x = 2$ are $(1, 1)$ $(2, 1)$ and the points of intersection of $y = x^2$ with $x = 1$, $x = 2$ are $(1, 1)$ $(2, 4)$

The region is indicated in the figure.

On changing the order we have from the figure y varying from 1 to 4 and $x = \sqrt{y}$ ($\because y = x^2$) to $x = 2$



$$\therefore \quad I = \int_{y=1}^{4} \int_{x=\sqrt{y}}^{2} (x^2 + y^2)\, dx\, dy$$

$$= \int_{y=1}^{4} \left[\frac{x^3}{3} + y^2 x \right]_{x=\sqrt{y}}^{2} dy$$

$$= \int_{y=1}^{4} \left\{ \left(\frac{8}{3} + 2y^2 \right) - \left(\frac{y^{3/2}}{3} + y^{5/2} \right) \right\} dy$$

$$= \frac{8}{3} [\, y \,]_1^4 + \frac{2}{3} [\, y^3 \,]_1^4 - \frac{1}{3} \cdot \frac{2}{5} [\, y^{5/2} \,]_1^4 - \frac{2}{7} [\, y^{7/2} \,]_1^4$$

$$= \frac{8}{3}(4 - 1) + \frac{2}{3}(64 - 1) - \frac{2}{15}(4^{5/2} - 1) - \frac{2}{7}(4^{7/2} - 1)$$

$$= 8 + 42 - \frac{2}{15}(32 - 1) - \frac{2}{7}(128 - 1)$$

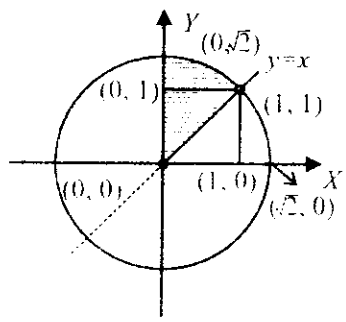
$$= 50 - \frac{62}{15} - \frac{254}{7} = \frac{1006}{105}$$

Thus $\quad I = \mathbf{1006 / 105}$

---

33. *Change the order of integration in* $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$ *and hence evaluate it.*

$$>> \quad I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$



$y = \sqrt{2 - x^2}$ or $x^2 + y^2 = 2$ is a circle with centre origin and radius $\sqrt{2}$. We shall find the points of intersection with the line $y = x$.

$$\therefore \quad 2x^2 = 2 \Rightarrow x = \pm 1$$

ie., (1 , 1) (− 1 , − 1) are the points of intersection. Since x varies from 0 to 1 (1 , 1) only is considered.

The region is shown in the figure.

The region on changing the order of integration is composed of two parts.

(i) y varying from 0 to 1 and x from 0 to y
(ii) y varying from 1 to $\sqrt{2}$ and x varying from 0 to $\sqrt{2-y^2}$.

$$\therefore \quad I = \int_{y=0}^{1} \int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \int_{y=1}^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

$$= \int_{y=0}^{1} \left[\sqrt{x^2+y^2}\right]_{x=0}^{y} dy + \int_{y=1}^{\sqrt{2}} \left[\sqrt{x^2+y^2}\right]_{x=0}^{\sqrt{2-y^2}} dy$$

$$= \int_{y=0}^{1} \left(\sqrt{2y^2}-y\right) dy + \int_{y=1}^{\sqrt{2}} (\sqrt{2}-y)\, dy$$

$$= (\sqrt{2}-1)\int_{0}^{1} y\, dy + \sqrt{2}\,[\,y\,]_1^{\sqrt{2}} - \left[\frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1 + \sqrt{2}\,(\sqrt{2}-1) - \left(1-\frac{1}{2}\right)$$

$$= (\sqrt{2}-1)\left(\frac{1}{2}\right) + 2 - \sqrt{2} - \frac{1}{2}$$

$$= \frac{1}{\sqrt{2}} - \sqrt{2} + 1 = \frac{\sqrt{2}-1}{\sqrt{2}}$$

Thus $I = 1-(1/\sqrt{2})$

---

34. *Show that* $\displaystyle\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} f(x,y)\, dy\, dx = \int_0^{a} \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\, dx\, dy$

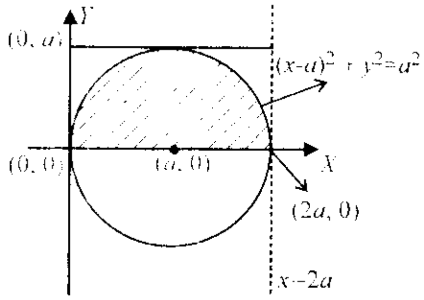
>> Here we need to only change the order of integration.

$$\text{L.H.S} = \int_{x=0}^{2a} \int_{y=0}^{\sqrt{2ax-x^2}} f(x,y)\, dy\, dx.$$

$$y = \sqrt{2ax-x^2} \quad \text{or} \quad y^2 = 2ax-x^2 \quad \text{or} \quad x^2+y^2-2ax = 0$$

ie., $(x-a)^2+(y-0)^2 = a^2$. This is a circle with centre $(a, 0)$ and radius a. It is evident that the circle passes through the origin having centre on the x-axis and radius equal to a. The region is as shown in the figure.

On changing the order y varies from 0 to a.



Also $(x-a)^2 = a^2-y^2 \Rightarrow (x-a) = \pm\sqrt{a^2-y^2}$

ie., $\quad x = a \pm \sqrt{a^2-y^2}$

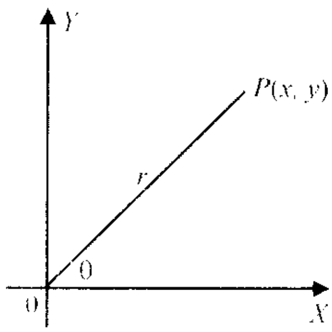
Thus we have $\displaystyle\int_{y=0}^{a}\int_{x=a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x, y)\,dx\,dy$ = R.H.S

Type-3. *Evaluation by changing into polars*

The principle involved is analogous to the evaluation of a definite integral by a suitable substitution. The method is explained in the article 5.24.

WORKED PROBLEMS

35. *Evaluate* $\displaystyle\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ *by changing to polar coordinates.*



>> In polars we have $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \quad x^2+y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$

Since x, y varies from 0 to ∞,
r also varies from 0 to ∞

In the first quadrant θ varies from 0 to $\pi/2$

Thus $\quad I = \displaystyle\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta.$

Put $\quad r^2 = t \;\therefore\; r\,dr = \dfrac{dt}{2}$; t also varies from 0 to ∞.

$$I = \int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta$$

4

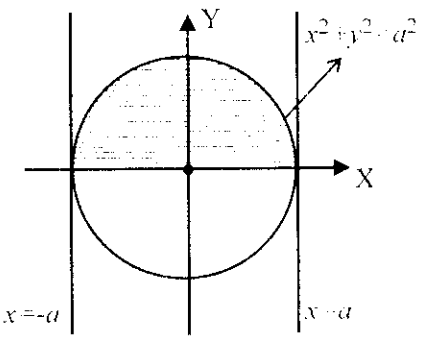
$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}[-e^{-t}]_{t=0}^{\infty}\,d\theta = -\frac{1}{2}\int_{\theta=0}^{\pi/2}(0-1)\,d\theta = \frac{1}{2}[\theta]_0^{\pi/2} = \frac{\pi}{4}$$

Thus $I = \pi/4$

---

36. *Change the integral* $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\sqrt{x^2+y^2}\,dy\,dx$ *into polars and hence evaluate the same.*

\>> The region of integration is as shown in the figure.



Clearly θ varies from 0 to π.

If $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

ie., $a^2 = r^2 \Rightarrow r = a$.

$\therefore$ r varies from 0 to a.

Also $dx\,dy = r\,dr\,d\theta$

$$\therefore \quad I = \int_{\theta=0}^{\pi}\int_{r=0}^{a} r\cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \frac{a^3}{3}[\theta]_0^{\pi} = \frac{a^3}{3}(\pi-0) = \frac{\pi a^3}{3}$$

Thus $I = \pi a^3/3$

---

37. *Evaluate* $\int_{0}^{a}\int_{0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\,dx\,dy$ *by changing into polars.*

\>> $$I = \int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\,dx\,dy$$

$x = \sqrt{a^2-y^2}$ or $x^2+y^2 = a^2$ is a circle with cenne origin and radius a. Since y varies from 0 to a the region of integration is the first quadrant of the circle.

In polars we have $x = r\cos\theta$, $y = r\sin\theta$ $\therefore$ $x^2+y^2 = r^2$

ie., $r^2 = a^2 \Rightarrow r = a$.

Also $x = 0$, $y = 0$ will give $r = 0$ and hence we can say that r varies from 0 to a.

In the first quadrant θ varies from 0 to $\pi/2$

Further we know that $dx\,dy = r\,dr\,d\theta$.

$$I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r \sin\theta \cdot r \cdot r\,dr\,d\theta = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^3 \sin\theta\,dr\,d\theta$$

$$I = \int_{r=0}^{a} r^3 [-\cos\theta]_0^{\pi/2} dr = \int_{r=0}^{a} -r^3 (0-1)\,dr = \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}$$

Thus $\boldsymbol{I = a^4/4}$

---

38. *If* $u = yz/x$, $v = zx/y$, $w = xy/z$, *Evaluate* $\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} du\,dv\,dw$ *by change of variable.*

>> We know that $du\,dv\,dw = J\,dx\,dy\,dz$ where J is the Jacobian of the given transformation.

We can obtain $J = 4$

$$\therefore \quad I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 4\,dz\,dy\,dx = 4 \int_{x=0}^{1} \int_{y=0}^{1-x} [z]_0^{1-x-y} dy\,dx$$

$$= 4 \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x-y)\,dy\,dx = 4 \int_{x=0}^{1} \left(y - xy - \frac{y^2}{2}\right)_{y=0}^{1-x} dx$$

$$= 4 \int_{x=0}^{1} \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2}\right] dx$$

$$= 2 \int_{x=0}^{1} [2(1-x) - 2x(1-x) - (1-x)^2]\,dx$$

$$= 2 \int_{x=0}^{1} (1 - 2x + x^2)\,dx = 2[x - x^2 + (x^3/3)]_0^1 = \frac{2}{3}$$

Thus $\boldsymbol{I = 2/3}$

Type-4. *Applications of double and triple integrals*

ILLUSTRATIVE PROBLEMS

1. *Find the area of the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *by double integration.*

Area $(A) = \iint_R dx\,dy.$

Referring to the figure in Problem - 14 we can write $A = 4A_1$ where A_1 is the area in the first quadrant and with reference to the same figure we have,

$$A = 4A_1 = 4\int_{x=0}^{a}\int_{y=0}^{(b/a)\sqrt{a^2-x^2}} dy\,dx = 4\int_{x=0}^{a}[y]_0^{(b/a)\sqrt{a^2-x^2}}\,dx$$

$$= 4\int_{x=0}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{x=0}^{a}$$

ie., $$= \frac{4b}{a}\left[0+\frac{a^2}{2}(\sin^{-1}1-\sin^{-1}0)\right] = \frac{4b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \pi ab$$

Thus the required area $(A) = \pi\, a\, b$ sq. units.

Note : *The area of the circle* $x^2+y^2 = a^2$ *by double integration is* πa^2.
This is a particular case of the example when $b = a$

---

2. *Find by double integration the area enclosed by the curve* $r = a(1+\cos\theta)$ *between* $\theta = 0$ *and* $\theta = \pi$

Area $A = \iint r\,dr\,d\theta$ where r varies from 0 to $a(1+\cos\theta)$ and θ from 0 to π

$$\therefore\quad A = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$A = \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{\theta=0}^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$A = \frac{a^2}{2}\int_{\theta=0}^{\pi}\left\{2\cos^2(\theta/2)\right\}^2 d\theta = 2a^2\int_0^{\pi}\cos^4(\theta/2)\,d\theta$$

Put $\theta/2 = \phi$, $d\theta = 2\,d\phi$ and ϕ varies from 0 to $\pi/2$

$$\therefore \quad A = 2a^2 \int_0^{\pi/2} \cos^4\phi \cdot 2\, d\phi$$

$$= 4a^2 \int_0^{\pi/2} \cos^4\phi\, d\phi = 4a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \text{ by the reduction formula.}$$

Thus the required area $A = 3\pi a^2/4$ sq. units.

---

3. *Find the volume of the tetrahedron bounded by the planes* $x = 0,\ y = 0,\ z = 0,\ x/a + y/b + z/c = 1$

>> $V = \iiint dx\, dy\, dz$

$x/a + y/b + z/c = 1 \quad \therefore\ z = c(1 - x/a - y/b)$

If $z = 0$, then $x/a + y/b = 1 \ \therefore\ y = b(1 - x/a)$

If $z = 0,\ y = 0$ then $x = a$

$$\therefore \quad V = \int_{x=0}^{a} \int_{y=0}^{b(1-x/a)} \int_{z=0}^{c(1-x/a-y/b)} dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx \quad \left(\because \int dz = z\right)$$

$$= c \int_{x=0}^{a} \left[y - \frac{x}{a} y - \frac{y^2}{2b} \right]_0^{b(1-x/a)} dx$$

$$= c \int_{x=0}^{a} \left\{ b\left(1 - \frac{x}{a}\right) - \frac{x}{a} b\left(1 - \frac{x}{a}\right) - \frac{b}{2}\left(1 - \frac{x}{a}\right)^2 \right\} dx$$

$$= c \int_{x=0}^{a} b\left(1 - \frac{x}{a}\right)\left\{1 - \frac{x}{a} - \frac{1}{2}\left(1 - \frac{x}{a}\right)\right\} dx$$

$$= c \int_{x=0}^{a} b\left(1 - \frac{x}{a}\right)\frac{1}{2}\left(1 - \frac{x}{a}\right) dx$$

$$= \frac{bc}{2} \int_{x=0}^{a} \left(1 - \frac{x}{a}\right)^2 dx = \frac{bc}{2}\left[\frac{-a}{3}\left(1 - \frac{x}{a}\right)^3\right]_{x=0}^{a}$$

$$V = -\frac{abc}{6}(0-1) = \frac{abc}{6}$$

Thus the required volume (V) = **abc/ 6** cubic units.

---

4. *Find the volume generated by the revolution of the cardioide* $r = a(1+\cos\theta)$ *about the initial line.*

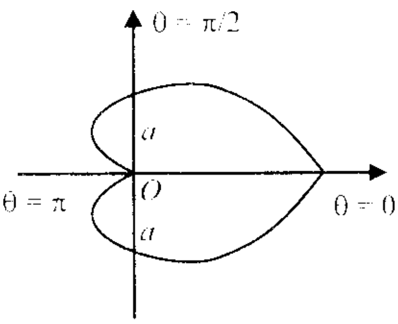
>> Volume of the solid of revolution in polars is given by $V = \iint_A 2\pi r^2 \sin\theta \, dr\, d\theta.$

Recollecting the nature and shape of the cardioide we have

$$V = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} 2\pi \left[\frac{r^3}{3}\right]_{r=0}^{a(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{2\pi}{3} \int_{\theta=0}^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta$$



Put $1+\cos\theta = t \;\therefore\; -\sin\theta\, d\theta = dt$

If $\theta = 0,\ t = 2;\ \theta = \pi,\ t = 0$

$$V = \frac{2\pi a^3}{3}\int_2^0 t^3(-dt) = \frac{2\pi a^3}{3}\int_0^2 t^3\, dt = \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2 = \frac{8\pi a^3}{3}$$

Thus the required volume (V) = $\mathbf{8\pi a^3/3}$ cubic units.

---

5. *A pyramid is bounded by three coordinate planes and the plane* $x+2y+3z = 6$*. Compute the volume by double integration.*

>> $V = \iint z\, dx\, dy$

Consider $x+2y+3z = 6$ or $x/6 + y/3 + z/2 = 1$

We have $z = 2[1-(x/6)-(y/3)]$

If $z = 0$, $(x/6)+(y/3) = 1 \Rightarrow y = 3[1-(x/6)]$

If $z = 0,\ y = 0,$ then $x = 6$

$$V = \int_{x=0}^{6}\int_{y=0}^{3[1-(x/6)]} 2[1-(x/6)-(y/3)]\, dy\, dx = 6, \text{ on evaluation.}$$

Thus the required volume (V) = **6** cubic units.

EXERCISES

Evaluate the following (1 to 5)

1. $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$

2. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{\sqrt{1+x^2+y^2}}$

3. $\int_0^a \int_0^{a-x} \int_0^{a-x-y} (x^2+y^2+z^2)\, dz\, dy\, dx$

4. $\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} x\, dx\, dy\, dz$

5. $\int_1^e \int_1^{\log y} \int_0^{e^x} \log z\, dz\, dy\, dx$

6. Evaluate $\iint_R xy^2\, dx\, dy$ over the region bounded by $y = x^2$, $y = 0$ and $x = 1$

7. Evaluate $\iint_R xy(x+y)\, dx\, dy$ taken over the region bounded by the parabolas $y^2 = x$ and $y = x^2$

8. Evaluate $\iint_R x^2 y\, dx\, dy$ over the region bounded by the curves $y = x^2$ and $y = x$

9. Evaluate $\iint_R xy\, dx\, dy$ where R is the region in the first quadrant bounded by the line $x + y = 1$

10. Evaluate $\iint_R x^2 y^2\, dx\, dy$ taken over the region bounded by the y-axis, x-axis and $x^2 + y^2 = 1$.

Evaluate the following by changing the order of integration (11 to 15)

11. $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\, dy\, dx$

12. $\int_0^a \int_0^{2\sqrt{ax}} x^2\, dx\, dy$

13. $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} (a-x)\,dy\,dx$

14. $\int_{0}^{a}\int_{\sqrt{ax}}^{a} \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}}$

15. $\int_{0}^{a}\int_{x^2/a}^{2a-x} xy\,dy\,dx$

ANSWERS

1. $3/8$
2. $\pi/4 \cdot \log(1+\sqrt{2})$
3. $a^5/20$
4. $\pi a^4/16$
5. $(e^2-8e+13)/2$
6. $1/24$
7. $3/28$
8. $1/35$
9. $1/6$
10. $\pi/96$
11. $a^3/28 + a/20$
12. $4a^4/7$
13. $\pi a^3/2$
14. $\pi a^2/6$
15. $3a^4/8$

5.3 Beta and Gamma functions

In this topic we define two special functions namely *Beta function* and *Gamma function* by means of an integral and study the associated properties. These help us to evaluate certain definite integrals which are either difficult or impossible to evaluate by various known methods of integration.

5.31 Definitions

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \ (m, n > 0) \qquad \ldots(1)$$

is called the **Beta function.**

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx, \ (n>0) \qquad \ldots(2)$$

is called the **Gamma function.**

These definitions can be put in the following alternative forms.

In (1) put $x = \sin^2\theta$. $dx = 2\sin\theta\cos\theta\,d\theta$